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TESIS

Bivariant K-theory of Generalized Weyl algebras

PARA OBTENER EL GRADO DE DOCTOR EN CIENCIAS CON MENCIÓN EN MATEMÁTICA

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Resumen

La K-teoría bivariante kk^{alg} en la categoría $\mathfrak{lca}$ de álgebras localmente convexas asigna grupos abelianos $kk_n^{\mathrm{alg}}(A,B)$, $n \in \mathbb{Z}$, a cada par de dichas algebras A y B y existen aplicaciones bilineales

$$kk_n^{\mathrm{alg}}(A,B) \times kk_m^{\mathrm{alg}}(B,C) \to kk_{n+m}^{\mathrm{alg}}(A,C)$$

para A, B y C álgebras localmente convexas y $m, n \in \mathbb{Z}$. Con este producto, podemos definir una categoría $\mathfrak{KK}^{\mathrm{alg}}$ cuyos objetos son álgebras localmente convexas y cuyos morfismos están dados por los grupos graduados $kk_*^{\mathrm{alg}}(A,B)$. De este modo, la K-teoría bivariante kk^{alg} se puede ver como un funtor $kk^{\mathrm{alg}}\colon \mathfrak{lca} \to \mathfrak{KK}^{\mathrm{alg}}$. Este funtor es universal con respecto a funtores *split* exactos, invariantes por diffotopías y $\mathcal{K}$-estables. En particular, un isomorfismo en $\mathfrak{KK}^{\mathrm{alg}}$ induce un isomorfismo en $\mathfrak{KK}^{\mathcal{L}_p}$ y en homología cíclica periódica bivariante HP.

En [10], se determina que los invariantes del álgebra de Weyl

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y | xy - yx = 1\rangle$$

son los mismos que los de $\mathbb{C}$. Esto es, se prueba que $A_1(\mathbb{C})$ es isomorfo a $\mathbb{C}$ en la categoria $\mathfrak{KK}^{\mathrm{alg}}$. En este trabajo, generalizamos el resultado a una familia de álgebras de Weyl generalizadas.

Como resultados del presente trabajo, calculamos la clase de isomorfismo en la categoría $\mathfrak{KK}^{\mathrm{alg}}$ de todas las álgebras de Weyl generalizadas no conmutativas $A = \mathbb{C}[h](\sigma, P)$, donde $\sigma(h) = qh + h_0$ es un automorfismo de $\mathbb{C}[h]$ y $P \in \mathbb{C}[h]$, excepto cuando $q \neq 1$ es una raíz de la unidad.

Abstract

The bivariant K-theory kk^{alg} in the category $\mathfrak{lca}$ of locally convex algebras asigns abelian groups $kk_n^{\text{alg}}(A,B)$, $n \in \mathbb{Z}$ to a pair A, B of such algebras and there are bilinear maps

$$kk_n^{\text{alg}}(A,B) \times kk_m^{\text{alg}}(B,C) \to kk_{n+m}^{\text{alg}}(A,C)$$

for every A, B and C locally convex algebras and $m, n \in \mathbb{Z}$. Using this product, we can define a category $\mathfrak{KK}^{\text{alg}}$ whose objects are locally convex algebras and whose morphisms are given by the graded groups $kk_*^{\text{alg}}(A,B)$. Then the bivariant K-theory kk^{alg} can be seen as a functor $kk^{\text{alg}}\colon \mathfrak{lca} \to \mathfrak{KK}^{\text{alg}}$. This functor is universal among split exact, diffotopy invariant and $\mathcal{K}$-stable functors. In particular, an isomorphism in $\mathfrak{KK}^{\text{alg}}$ induces an isomorphism in $\mathfrak{KK}^{\mathcal{L}_p}$ and in bivariant periodic cyclic homology HP.

In [10], the invariants of the Weyl algebra

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y | xy - yx = 1\rangle$$

are determined to be the same as those of $\mathbb{C}$. That is, $A_1(\mathbb{C})$ is isomorphic to $\mathbb{C}$ in the category $\mathfrak{KK}^{\text{alg}}$. In the present work, we generalize this result to a family of generalized Weyl algebras.

As results, we compute the isomorphism class in the category $\mathfrak{KK}^{\text{alg}}$ of all non commutative generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$, where $\sigma(h) = qh + h_0$ is an automorphism of $\mathbb{C}[h]$ and $P \in \mathbb{C}[h]$, except when $q \neq 1$ is a root of unity.

Introduction

In [10], Cuntz defined a bivariant K-theory kk^{alg} in the category $\mathfrak{lca}$ of locally convex algebras. These are algebras A that are complete locally convex vector spaces over $\mathbb{C}$ with a jointly continuous multiplication $\cdot\colon A \times A \to A$. To a pair of locally convex algebras A and B, there correspond abelian groups $kk_n^{\mathrm{alg}}(A, B)$, $n \in \mathbb{Z}$ and there are bilinear maps

$$kk_n^{\mathrm{alg}}(A, B) \times kk_m^{\mathrm{alg}}(B, C) \to kk_{n+m}^{\mathrm{alg}}(A, C)$$

for every A, B and C locally convex algebras and $m, n \in \mathbb{Z}$. Using this product, we can define a category $\mathfrak{KK}^{\mathrm{alg}}$ whose objects are locally convex algebras and whose morphisms are given by the graded groups $kk_*^{\mathrm{alg}}(A, B)$. Then the bivariant K-theory kk^{alg} can be seen as a functor $kk^{\mathrm{alg}}\colon \mathfrak{lca} \to \mathfrak{KK}^{\mathrm{alg}}$. This functor is universal among split exact, diffotopy invariant and $\mathcal{K}$-stable functors:

Theorem 0.1. *[Theorem 7.26 in* [11] *] If F is a covariant functor from the category of bornological algebras to an abelian category $\mathfrak{C}$ that is diffotopy invariant, half exact for linearly split extensions and $\mathcal{K}$-stable then $F = \bar{F} \circ kk^{\mathrm{alg}}$ for a unique homological functor $\bar{F}\colon \mathfrak{KK}^{\mathrm{alg}} \to \mathfrak{C}$.*

This property implies the existence of a bivariant Chern-Connes character to bivariant periodic cyclic homology, i.e. for any pair of locally convex algebras A and B, there are natural maps $ch\colon kk_n^{\mathrm{alg}}(A, B) \to HP_n(A, B)$ that commute with the products of kk^{alg} and of HP. In particular, an isomorphism in $\mathfrak{KK}^{\mathrm{alg}}$ induces an isomorphism in bivariant periodic cyclic homology HP.

The coefficient ring $kk_0^{\mathrm{alg}}(\mathbb{C}, \mathbb{C})$ has not been computed. However, the coefficient ring can be computed for a related bivariant K-theory. In [12], Cuntz and Thom define

$kk_n^{\mathcal{L}_p}(A,B) = kk_n^{\text{alg}}(A, B\otimes_\pi \mathcal{L}_p)$ where $\mathcal{L}_p \subseteq B(\mathbb{H})$ is the p-th Schatten ideal. In the same article, they prove that $kk_0^{\mathcal{L}_p}(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ and $kk_1^{\mathcal{L}_p}(\mathbb{C},\mathbb{C}) = 0$. The functor $kk^{\mathcal{L}_p}$ satisfies the conditions of Theorem 0.1, thus there is a functor $\mathfrak{KK}^{\text{alg}} \to \mathfrak{KK}^{\mathcal{L}_p}$.

The category of locally convex algebras includes all algebras with a countable basis over $\mathbb{C}$ with the topology given by all seminorms. The Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\langle x, y | xy - yx = 1\rangle$ is one of such algebras and in [10] it is proven that $A_1(\mathbb{C})$ is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\text{alg}}$. By the universal property of $\mathfrak{KK}^{\text{alg}}$ this implies $\mathfrak{KK}_0^{\mathcal{L}_p}(\mathbb{C}, W) = \mathbb{Z}$ and $\mathfrak{KK}_1^{\mathcal{L}_p}(\mathbb{C}, W) = 0$.

The results from [10] together with those obtained for $\mathbb{Z}$-graded C*-algebras in [21], [20], [13] and [1] motivate the construction of tools for finding the invariants of other $\mathbb{Z}$-graded locally convex algebras.

Tools for computing the K-theory of $\mathbb{Z}$-graded C*-algebras date back to the Pimsner-Voiculescu sequence (see [21]). Let A be a C*-algebra and $\alpha \in \text{Aut}(A)$. The crossed product $A \rtimes_\alpha \mathbb{Z}$ is the universal C*-algebra generated by A and a unitary element u satisfying the relation

$$ua = \alpha(a)u$$

for all $a \in A$. The $\mathbb{Z}$-grading is defined by setting the degree of u equal to 1 and the degree of all elements of A equal to 0. The Pimsner-Voiculescu sequence is a classical result for computing the K-theory of a crossed product by $\mathbb{Z}$.

Theorem 0.2 (Theorem 2.4 in [21])**.** *There is an exact sequence*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{1-\alpha_*} & K_1(A). \end{array}$$

There are results that generalize the Pimsner-Voiculescu exact sequence for other $\mathbb{Z}$-graded C*-algebras such as Cuntz-Pimsner algebras defined in [20], covariance algebras associated to partial automorphisms (see [13]) and for generalized crossed products (see [1]).

A similar result can be obtained for smooth crossed products $A\hat{\rtimes}_\alpha\mathbb{Z}$, where A is a locally convex algebra and $\alpha \in \text{Aut}(A)$ (see [10]). The smooth crossed product $A\hat{\rtimes}_\alpha\mathbb{Z}$ is defined as the universal locally convex algebra generated by A together with an invertible element u satisfying $uxu^{-1} = \alpha(x)$ for all $x \in A$. In this case we have the following theorem.

Theorem 0.3 (Theorem 14.3 in [10])**.** *For any locally convex algebra D, there is an exact sequence*

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(D,A) & \xrightarrow{\cdot(1-kk(\alpha))} & kk_0^{\mathrm{alg}}(D,A) & \longrightarrow & kk_0^{\mathrm{alg}}(D,A\hat{\rtimes}_\alpha\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ kk_1^{\mathrm{alg}}(D,A\hat{\rtimes}_\alpha\mathbb{Z}) & \longleftarrow & kk_1^{\mathrm{alg}}(D,A) & \xleftarrow{\cdot(1-kk(\alpha))} & kk_1^{\mathrm{alg}}(D,A). \end{array}$$

The locally convex algebra analog to a generalized crossed product is called a smooth generalized crossed product and defined in [15].

Definition 0.4. A gauge action γ on a locally convex algebra B is a pointwise continuous action of S^1 on B. An element $b \in B$ is called gauge smooth if the map $t \mapsto \gamma_t(b)$ is smooth.

If we have a gauge action on B, then $B_n = \{b \in B | \gamma_t(b) = t^n b,\ \forall t \in S^1\}$ define a natural $\mathbb{Z}$-grading of B.

Definition 0.5. A smooth generalized crossed product is a locally convex algebra B with an involution and a gauge action such that

- B_0 and B_1 generate B as a locally convex involutive algebra.
- all b are gauge smooth and the map $B \to C^\infty(S^1, B)$ is continuous.

In [15], 6-term exact sequences for smooth generalized crossed products B that satisfy the condition of being tame smooth are constructed (see definition 18 in [15]). These sequences relate the kk^{alg} invariants of B with the kk^{alg} invariants of the degree 0 subalgebra B_0.

Theorem 0.6 (Theorem 36 in [15])**.** *Let B be a tame smooth generalized crossed product. For any locally convex algebra D we have a 6-term exact sequence*

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(D,B_0) & \longrightarrow & kk_0^{\mathrm{alg}}(D,B_0) & \longrightarrow & kk_0^{\mathrm{alg}}(D,B) \\ \uparrow & & & & \downarrow \\ kk_1^{\mathrm{alg}}(D,B) & \longleftarrow & kk_1^{\mathrm{alg}}(D,B_0) & \longleftarrow & kk_1^{\mathrm{alg}}(D,B_0), \end{array}$$

and a similar sequence on the other variable.

In this thesis, we study a family of generalized Weyl algebras.

Definition 0.7. Let D be a ring, $\sigma \in \mathrm{Aut}(D)$ and a a central element of D. The generalized Weyl algebra $D(\sigma, a)$ is the algebra generated by x and y over D satisfying

$$xd = \sigma(d)x, \ yd = \sigma^{-1}(d)y, \ yx = a \text{ and } xy = \sigma(a) \tag{0.1}$$

for all $d \in D$.

We compute the isomorphism class in $\mathfrak{K}\mathfrak{K}^{\mathrm{alg}}$ of all non commutative generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$, where $\sigma(h) = qh + h_0$ is an automorphism of $\mathbb{C}[h]$ and $P \in \mathbb{C}[h]$, except when $q \neq 1$ is a root of unity. In the table below we list all possible cases for A and our results.

Conditions		Results	
P is constant	$P = 0$	$A \cong_{\mathfrak{K}\mathfrak{K}^{\mathrm{alg}}} \mathbb{C}$	Prop 4.20
	$P \neq 0$	$A \cong_{\mathfrak{K}\mathfrak{K}^{\mathrm{alg}}} S\mathbb{C} \oplus \mathbb{C}$	Prop 4.19
P is nonconstant with r distinct roots	q not a root of unity	$A \cong_{\mathfrak{K}\mathfrak{K}^{\mathrm{alg}}} \mathbb{C}^r$	Thm 4.13 Prop 4.17
	$q = 1$ and $h_0 \neq 0$	$A \cong_{\mathfrak{K}\mathfrak{K}^{\mathrm{alg}}} \mathbb{C}^r$	Thm 4.13
	$q \neq 1$, a root of unity	No result	
	$q = 1$ and $h_0 = 0$	No result	

Generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ are locally convex algebras when given the fine topology. They are $\mathbb{Z}$-graded with a grading defined by $\deg y = 1$ and $\deg x = -1$. There is an action of S^1 defined by $\gamma_t(\omega_n) = t^n \omega_n$ for $\omega_n \in A_n$. When $P \in \mathbb{R}[h]$ and q and h_0 are real, they have an involution defined by $y^* = x$, $x^* = y$ and d^* defined by conjugating all coefficients of d, for $d \in \mathbb{C}[h]$. Generalized Weyl algebras over $\mathbb{C}[h]$ satisfying these conditions are smooth generalized crossed products that are tame smooth if and only if P is a non zero constant polynomial (see Remark 3.13).

Hence, if $P \in \mathbb{C}[h]$ is a non-constant polynomial we cannot use the results of [15]. However, in most cases we can construct an explicit faithful representation of A, which allows us to follow the general strategy of [10] and [15], in order to determine the $\mathfrak{KK}^{\mathrm{alg}}$ class of A.

Our main result is Theorem 4.13, which computes the isomorphism class of A in $\mathfrak{KK}^{\mathrm{alg}}$ in the following two cases:

- $q = 1$ and $h_0 \neq 0$.
- q is not a root of unity and P has a root different from $\frac{h_0}{q-1}$.

In each of these cases we construct an exact triangle

$$SA \to A_1A_{-1} \xrightarrow{0} A_0 \to A, \tag{0.2}$$

in the triangulated category $(\mathfrak{KK}^{\mathrm{alg}}, S)$, where A_n is the subspace of degree n of the $\mathbb{Z}$-graded algebra A (see Lemma 3.3). In order to construct the exact triangle in (0.2) we follow the methods of [15]: we construct a linearly split extension

$$0 \to \Lambda_A \to \mathcal{T}_A \to A \to 0$$

and prove

$$\mathcal{T}_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0 \quad \text{and} \quad \Lambda_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_1A_{-1}.$$

The exact triangle in (0.2) yields an isomorphism $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0 \oplus S(A_1A_{-1})$. The main result now follows after we prove $A_1A_{-1} \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C}^{r-1}$ in Proposition 4.12, since $A_0 = \mathbb{C}[h] \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}$.

The main result allows for the computation of the isomorphism class in $\mathfrak{KK}^{\mathrm{alg}}$ of the quantum Weyl algebra, the primitive factors B_λ of $U(\mathfrak{sl}_2)$ and the quantum weighted projective lines $\mathcal{O}(\mathbb{WP}_q(k,l))$ (see [4]).

For the sake of completeness, we also discuss the case where P is a constant polynomial or has only $\frac{h_0}{1-q}$ as a root.

In the case where $A = \bigoplus_{n\in\mathbb{N}} A_n$ is an $\mathbb{N}$-graded locally convex algebra with the fine topology, it can be shown that $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0$ (see Lemma 4.16). This is the case when

- P is nonconstant, q is not a root of unity and P has only $\frac{h_0}{q-1}$ as a root or
- $P = 0$.

In these cases we obtain $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}$.

In the case where P is a nonzero constant polynomial, we follow the construction of [15]. In this case, there is an exact triangle

$$SA \to A_0 \xrightarrow{0} A_0 \to A, \tag{0.3}$$

in the triangulated category $(\mathfrak{KK}^{\mathrm{alg}}, S)$ and we obtain $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C} \oplus \mathbb{C}$.

In the case where $q = 1$ and $h_0 = 0$, we have $\sigma = \mathrm{id}$ and so $A \cong \mathbb{C}[h, x, y]/(xy - P)$ is a commutative algebra. This case and the case where $q \neq 1$ is a root of unity remain open.

This thesis is organized as follows. In Chapter 1, we recall basic results on locally convex algebras. Lemma 1.38 is a technical result which asserts that the projective tensor product of the Toeplitz algebra $\mathcal{T}$ with an algebra with a countable basis over $\mathbb{C}$ is the algebraic tensor product. In Chapter 2 we recall the definition and properties of kk^{alg} following [10] and [12]. In Chapter 3, we define generalized Weyl algebras and construct explicit faithful representations when $q = 1$ and $h_0 \neq 0$, and when q is not a root of unity and P has a root different from $\frac{h_0}{q-1}$. In Chapter 4, we compute the isomorphism class in $\mathfrak{KK}^{\mathrm{alg}}$ of all noncommutative generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ where $\sigma(h) = qh + h_0$ except when $q \neq 1$ is a root of unity.

Chapter 1

Locally convex algebras

In this chapter we discuss the category of locally convex algebras $\mathfrak{lca}$, differential homotopies, linearly split extensions of locally convex algebras and some important locally convex algebras which we use for the definition of kk^{alg} and for computations. We follow the discussions of [15], [11] and [10].

1.1 Locally convex algebras

We begin with the definition of the category of locally convex algebras. This category is broad enough to contain all algebras with a countable basis over $\mathbb{C}$. In what follows we consider only vector spaces and algebras over $\mathbb{C}$.

Definition 1.1. A locally convex algebra A is a complete locally convex vector space over $\mathbb{C}$ which is an algebra such that for any continuous seminorm p in A there is a continuous seminorm q in A such that $p(ab) \leq q(a)q(b)$ for all $a, b \in A$. This is equivalent to saying that the multiplication is jointly continuous.

Definition 1.2. Let A be a locally convex algebra. A seminorm p of A is called sub multiplicative if $p(ab) \leq p(a)p(b)$, for all $a, b \in A$. If the topology of A can be defined by a family of submultiplicative seminorms we say that A is an m-algebra.

The category $\mathfrak{lca}$ has locally convex algebras as objects and the morphisms are continuous homomorphisms.

Examples 1.3. The following are examples of locally convex algebras.

1. All algebras with a countable basis over $\mathbb{C}$. These are locally convex algebras when the topology is generated by all seminorms (Proposition 2.1 on [10]). The Weyl algebra and generalized Weyl algebras (defined in Chapter 3) are examples of this kind of algebras.

2. $\mathcal{C}^\infty([0,1])$ is a locally convex algebra with the topology defined by the family of seminorms
$$p_n(f) = ||f|| + ||f'|| + \frac{1}{2}||f''|| + \cdots + \frac{1}{n!}||f^{(n)}||$$
where $||f|| = sup\{f(t)|t \in [0,1]\}$.

3. More generally, $\mathcal{C}^\infty(M)$ is a locally convex algebra for any compact manifold M.

4. The smooth Toeplitz algebra $\mathcal{T}$ and the algebra of smooth compact operators $\mathcal{K}$ (to be defined in Section 1.4).

There are several possible ways to topologize the tensor product $V \otimes W$ of two locally convex vector spaces. These different topologies will lead to different completions. The two most common completions are the projective completion $V \otimes_\pi W$ and the equicontinuous completion $V \otimes_\epsilon W$. For locally convex algebras, we use the projective completion.

Definition 1.4. Let V and W be locally convex spaces. We define the projective tensor product $V \otimes_\pi W$ of V and W as the completion of $V \otimes W$ with respect to the family of seminorms
$$(p \otimes q)(z) = \inf\{\sum_{i=1}^{n} p(a_i)q(b_i) \mid z = \sum_{i=1}^{n} a_i \otimes b_i, n \geq 1, a_i \in A, b_i \in B\}$$
where p and q are continuous seminorms on V and W respectively.

Lemma 1.5. *If A and B are locally convex algebras, then $A \otimes_\pi B$ is a locally convex algebra.*

Proof. We will proof that the multiplication in $A \otimes B$ is continuous with respect to the family of seminorms $p \otimes q$. Let $z = z_1 z_2 \in A \otimes B$. Let $p \otimes q$ be a continuous seminorm on

$A \otimes B$ and $\bar{p}$, $\bar{q}$ be such that for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have $p(a_1 a_2) \leq \bar{p}(a_1)\bar{p}(a_2)$ and $q(b_1 b_2) \leq \bar{q}(b_1)\bar{q}(b_2)$. Then for all $\epsilon > 0$ we have expressions $z_1 = \sum_i a_i \otimes b_i$ and $z_2 = \sum_j c_j \otimes d_j$ such that

$$\begin{aligned}\sum_i \bar{p}(a_i)\bar{q}(b_i) &\leq (\bar{p} \otimes \bar{q})(z_1) + \epsilon \\ \sum_j \bar{p}(c_j)\bar{q}(d_j) &\leq (\bar{p} \otimes \bar{q})(z_2) + \epsilon.\end{aligned}$$

Now we have $z = \sum_{i,j} a_i c_j \otimes b_i d_j$ and

$$\begin{aligned}(p \otimes q)(z) &\leq \sum_{i,j} p(a_i c_j) q(b_i d_j) \\ &\leq \sum_{i,j} \bar{p}(a_i)\bar{p}(c_j)\bar{q}(b_i)\bar{q}(d_j) \\ &\leq [(\bar{p} \otimes \bar{q})(z_1) + \epsilon][(\bar{p} \otimes \bar{q})(z_2) + \epsilon]\end{aligned}$$

therefore $(p \otimes q)(z) \leq [(\bar{p} \otimes \bar{q})(z_1)][(\bar{p} \otimes \bar{q})(z_2)]$. Since $A \otimes B$ is dense in $A \otimes_\pi B$, the product in $A \otimes_\pi B$ is also continuous. □

Remark 1.6. The completions $V \otimes_\pi W$ and $V \otimes_\epsilon W$ coincide when either V or W is a nuclear space (see Definition 50.1 and Theorem 50.1 in [23]). The main example of an infinite dimensional nuclear space is the space of rapidly decreasing sequences.

Definition 1.7. Define $\mathfrak{s}$ to be the space of rapidly decreasing sequences of complex numbers. These are sequences $a = (a_i)_{i \in \mathbb{N}}$ such that the sums

$$p_n(a) = \sum_{i=0}^{\infty} |1 + i|^n |a_i|$$

are finite for all $n \in \mathbb{N}$. The locally convex topology is defined by the seminorms p_n.

Example 1.8. We define $\mathbb{C}[0,1]$ as the (closed) subalgebra of $\mathcal{C}^\infty([0,1])$ of functions with all derivatives vanishing at 0 and 1. Since $\mathcal{C}^\infty([0,1])$ is a nuclear space and $\mathbb{C}[0,1]$ is a linear subspace, then $\mathbb{C}[0,1]$ is nuclear (see item (50.3) in Proposition 50.1 in [23]). Therefore, for any locally convex algebra A, $\mathbb{C}[0,1] \otimes_\pi A = A[0,1]$, the algebra of $\mathcal{C}^\infty$ functions with values in A and all derivatives vanishing at 0 and 1. We define $A[0,1)$ and $A(0,1)$ as the subalgebras of $A[0,1]$ that consist of functions that vanish at 1, and at 0 and 1 respectively.

Definition 1.9. We define SA and CA to be the algebras $A(0,1)$ and $A[0,1)$ and we call them the suspension and the cone of A respectively.

Definition 1.10. There is an extension

$$0 \to SA \to CA \to A \to 0.$$

We name this extension the cone extension of A. We will see that this is a linearly split extension as defined in 1.3.

Note that $S\colon \mathfrak{lca} \to \mathfrak{lca}$ is a functor. Given a morphism ϕ of locally convex algebras $\mathbb{C}[0,1]$, there is a morphism $S(\phi)\colon SA \to SB$ defined by $f \mapsto \phi \circ f$. We can iterate this functor n times to obtain $S^n A$ and $S^n(f)$.

1.2 Diffotopies

The bivariant K-theory kk^{alg} is invariant with respect to differentiable homotopies also known as diffotopies. The reader can consult section 6.1 in [11] for more details on diffotopies.

Definition 1.11. Let $\phi_0, \phi_1\colon A \to B$ be morphisms of locally convex algebras. A diffotopy between ϕ_0 and ϕ_1 is a morphism $\Phi\colon A \to \mathcal{C}^\infty([0,1], B)$ such that $\mathrm{ev}_i \circ \Phi = \phi_i$. If there is a diffotopy between ϕ_0 and ϕ_1 we call them diffotopic and write $\phi_0 \simeq \phi_1$.

Using a reparameterization of the interval we can assume that all derivatives of Φ at 0 and 1 vanish and therefore we can assume that a diffotopy is given by a morphism $\Phi\colon A \to B[0,1]$. With this characterization we can define a concatenation of diffotopies and therefore show that diffotopy is an equivalence relation.

Remark 1.12. The existence of a diffotopy $\Phi\colon A \to B[0,1]$ implies the existence of a family of homomorphisms $\phi_t\colon A \to B$ such that $t \to \phi_t(x)$ is in $B[0,1]$ for each $x \in A$. However, as it is proven in [14], the existence of such a family is not equivalent to the existence of a diffotopy between ϕ_0 and ϕ_1 because Φ might fail to be continuous. However, Φ will be continuous when A and B are Frechet (because of the Closed Graph Theorem) or when

A has the topology defined by all seminorms. We use this fact to justify the existence of diffotopies in Lemmas 4.16 and 1.36.

Definition 1.13. Given $F_0, F_1 \colon A \to B[0,1]$ their concatenation is the continuous homomorphism

$$F_0 \bullet F_1(a)(t) = \begin{cases} F_0(a)(2t) & ,0 \leq t \leq 1/2 \\ F_1(a)(2t-1) & ,1/2 \leq t \leq 1 \end{cases}$$

Definition 1.14. Given two locally convex algebras A and B, we denote by $\langle A, B \rangle$ the set of diffotopy clases of continuous homomorphisms from A to B. We denote by $\langle \phi \rangle$ the diffotopy class of a continuous homomorphism $\phi \colon A \to B$.

Lemma 1.15. *There is a group structure in $\langle A, SB \rangle$ given by concatenation. The group stuctures in $\langle A, S^n B \rangle$ that we get from concatenation in different variables all agree and are abelian for $n \geq 2$.*

Proof. See Lemma 6.4 in [11]. □

Next we define contractible locally convex algebras.

Definition 1.16. A locally convex algebra A is called contractible if the identity map is diffotopic to 0.

Examples 1.17. Examples of contractible locally convex algebras are $t\mathbb{C}[t]$ and CA. The diffotopies are given by $\phi_s \colon t\mathbb{C}[t] \to t\mathbb{C}[t]$, $\phi_s(t) = st$ and $\psi_s \colon CA \to CA$, $\psi_s(f)(t) = f(st)$, respectively. Note that the algebras $(t - t_0)\mathbb{C}[t]$ are isomorphic to $t\mathbb{C}[t]$ and therefore are also contractible.

We conclude this section with a note on $\mathbb{N}$-graded algebras.

Lemma 1.18. *Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be an $\mathbb{N}$-graded locally convex algebra with the fine topology, then A is diffotopy equivalent to A_0.*

Proof. The diffotopy is given by the family of morphisms $\phi_t \colon A \to A$, $t \in [0,1]$, sending an element $a_n \in A_n$ to $t^n a_n$. When $t = 1$ we recover the identity and when $t = 0$ the morphism is a retraction of A onto A_0. □

This lemma will be useful for computing the invariants of a particular family of generalized Weyl algebras (see Section 4).

1.3 Extensions of locally convex algebras

In this section, we define linearly split extensions of locally convex algebras and their classifying maps. Extensions play a key role in the definition of kk^{alg} because we can characterize the suspension stable category in terms of extensions of locally convex algebras of arbitrary length using their classifying maps.

Definition 1.19. An extension of locally convex algebras

$$0 \to I \to E \to B \to 0$$

is linearly split if there is a continuous linear section $s\colon B \to E$. Similarly we define extensions of length n to be chain complexes

$$0 \to I \to E_n \to \cdots \to E_1 \to B \to 0$$

and we say an extension of lenght n is linearly split if there is a continuous linear maps of degree -1 such that $ds + sd = \mathrm{id}$ (where d is the differential of the chain complex).

Example 1.20. Let A be a locally convex algebra. The cone extension of A

$$0 \to SA \to CA \to A \to 0$$

is a linearly split extension. There exists a continuous linear section $s\colon A \to CA$ defined by $a \in A \mapsto f \in CA$ with $f(t) = (1 - \psi(t))a$, where $\psi\colon [0,1] \to [0,1]$ is a C^∞ bijection with $f(0) = 0$, $f(1) = 1$ and all derivatives vanishing at 0 and at 1.

Now, we define the tensor algebra which has a universal property in the category of locally convex algebras. It is a completion of the usual algebraic tensor algebra. Let V be a complete locally convex vector space. The algebraic tensor algebra is defined as

$$T_{\mathrm{alg}}V = \bigoplus_{n=1}^{\infty} V^{\otimes n}.$$

Notice that we are considering a non-unital algebraic tensor algebra. We will topologize $T_{\text{alg}}V$ with the following family of seminorms. First notice that there is a linear map $\sigma\colon V \to T_{\text{alg}}V$ mapping V into the first summand. Consider all seminorms of the form $\alpha \circ \phi$, where ϕ is a homomorphism from $T_{\text{alg}}V$ into a locally convex algebra B such that $\phi \circ \sigma$ is continuous on V and α is a continuous seminorm on B.

Definition 1.21. The tensor algebra TV is the completion of $T_{\text{alg}}V$ with respect to the family of seminorms $\{\alpha \circ \phi\}$ defined above.

Proposition 1.22. *The tensor algebra TV is a locally convex algebra.*

Proof. First we will show that the multiplication in $T_{\text{alg}}V$ is continuous. Let $x, y \in T_{\text{alg}}V$. With α, ϕ and σ as above, we have $(\alpha \circ \phi)(xy) = \alpha(\phi(x)\phi(y))$. Since B is a locally convex algebra, there exists a continuous seminorm β in B such that $\alpha(b_1 b_2) \leq \beta(b_1)\beta(b_2)$ for all $b_1 b_2 \in B$. Therefore, we have $(\alpha \circ \phi)(xy) \leq (\beta \circ \phi)(x)(\beta \circ \phi(y))$. Since TV is the completion of $T_{\text{alg}}V$, the multiplication in TV is also continuous. □

The tensor algebra satisfies the following universal property.

Proposition 1.23. *Given a continuous linear map $s\colon V \to B$ from a complete locally convex vector space V to a locally convex algebra B, there is a unique morphism of locally convex algebras $\tau\colon TV \to B$ such that $\tau \circ \sigma = s$. The morphism τ is defined by $\tau(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\ldots s(x_n)$ where $x_i \in V$.*

Proof. By the universal property of $T_{\text{alg}}V$, we have a unique morphism $\phi\colon T_{\text{alg}}V \to B$ such that $\phi \circ \sigma = s$. The morphism ϕ is continuous because s is continuous and for every continuous seminorm α in B, $\alpha \circ \phi$ is a continuous seminorm in $T_{\text{alg}}V$ (see Definition 1.21). Since B is complete, ϕ extends to a morphism $\tau\colon TV \to B$. Two morphisms $\tau_1, \tau_2\colon TV \to B$ that satisfy $\tau_1 \circ \sigma = \tau_2 \circ \sigma = s$ coincide in $T_{\text{alg}}V$ and therefore are the same. □

In particular, if A is a locally convex algebra, the identity map $\text{id}\colon A \to A$ induces a morphism $\pi\colon TA \to A$.

We use the universal property of TA to construct a universal extension. There is an extension

$$0 \longrightarrow JA \longrightarrow TA \xrightarrow{\pi} A \longrightarrow 0,$$

where JA is defined as the kernel of $\pi\colon TA \to A$, which has a canonical linear section $\sigma\colon A \to TA$. This extension is universal in the sense that given any extension of locally convex algebras $0 \to I \to E \to B \to 0$ with lineal split s and a morphism $\alpha\colon A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & JA & \longrightarrow & TA & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where $\tau\colon TA \to E$ is the morphism induced by the continuous linear map $s \circ \alpha\colon A \to E$ and $\gamma\colon JA \to I$ is the restriction of τ.

Notice that $J\colon \mathfrak{lca} \to \mathfrak{lca}$ is a functor. Given a morphism $\alpha\colon A \to B$, consider the extension $0 \to JB \to TB \to B \to 0$ with its cannonical continuous linear section. Then we define $J(\alpha)\colon JA \to JB$ in the natural way. We can iterate this construction n times to obtain J^nA and $J^n(\alpha)$.

We observe that the map $\gamma\colon JA \to I$ is unique up to diffotopy. Given two linear sections s_1 and s_2 then $s_t = ts_1 + (1-t)s_2$ is a smooth family of linear sections which induces a diffotopy γ_t.

Definition 1.24. The morphism $\gamma\colon JA \to I$ is called the classifying map of both the extension $0 \to I \to E \to B \to 0$ and the morphism $\alpha\colon A \to B$. It is well-defined up to diffotopy.

Similarly, we can define the classifying map of an extension

$$0 \to I \to E_n \to \cdots \to E_1 \to B \to 0$$

and a morphism $\alpha\colon A \to B$ to be the map $\gamma\colon J^nA \to I$ in

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & J^nA & \longrightarrow & T(J^{n-1}A) & \longrightarrow & \cdots & \longrightarrow & TA & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & I & \longrightarrow & E_n & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

It will also be unique up to diffotopy.

Examples 1.25.

1. For any locally convex algebra A, there is a classifying map $JA \to SA$ associated to its cone extension $0 \to SA \to CA \to A \to 0$.

2. Given a morphism $\alpha\colon J^n A \to S^m B$, there is an induced morphism $\alpha'\colon J^{n+1}A \to S^{m+1}B$ defined up to diffotopy by the morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^{n+1}A & \longrightarrow & T(J^n A) & \longrightarrow & J^n A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & S^{m+1}B & \longrightarrow & C(S^m B) & \longrightarrow & S^m B & \longrightarrow & 0 \end{array}$$

Finally, we study the interplay between the functors J and S. We define a natural projection from $J^j S^i B$ onto $S^i J^j B$.

Definition 1.26. For a locally convex algebra B and $i, j \in \mathbb{N}$, we define $\kappa_B^{i,j}$ to be the classifying map in the extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^j S^i B & \longrightarrow & \cdots & \longrightarrow & S^i B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \kappa} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & S^i J^j B & \longrightarrow & \cdots & \longrightarrow & S^i B & \longrightarrow & 0 \end{array}$$

where the bottom sequence is obtained by tensoring the sequence

$$0 \to J^j B \to T(J^{j-1}B) \to \cdots \to B \to 0$$

with $S^i\mathbb{C} = \mathbb{C}(0,1)^i$.

1.4 The algebra of smooth compact operators and the smooth Toeplitz algebra

We will define kk^{alg} to be stable with respect to the algebra $\mathcal{K}$ of smooth compact operators. This algebra will play a role analogue to the one of $\mathbb{K}$, the algebra of compact operators used in Kasparov's KK-theory.

We will also define the smooth Toeplitz algebra $\mathcal{T}$. This algebra will be the locally convex algebra analogue to $\mathcal{T}_{C^*}$, the Toeplitz C*-algebra. It will be used to prove results such as Bott periodicity, Pimsner-Voiculescu exact sequences and the sequences we will construct for generalized Weyl algebras.

An important result of this section is the diffotopy of Lemma 1.36. This diffotopy is a smooth version of a classical homotopy of C*-algebras given by Cuntz in [6].

Definition 1.27. The algebra of smooth compact operators $\mathcal{K}$ is defined as the algebra of $\mathbb{N} \times \mathbb{N}$ matrices $a = (a_{ij})_{i,j\in\mathbb{N}}$ such that $q_n(a) = \sum_{i,j\in\mathbb{N}}(1+i+j)^n|a_{i,j}|$ is finite for all $n \in \mathbb{N}$. The topology is defined by the seminorms q_n.

Let $\mathbb{H} = l^2(\mathbb{N})$ be the Hilbert space with a countable basis. We can define $\mathbb{K} \subseteq B(\mathbb{H})$, the algebra of compact operators, as the closure of the subalgebra of finite rank operators. The algebra $\mathbb{K}$ can also be viewed as $\mathbb{N} \times \mathbb{N}$ matrices with square summable coefficients; that is, matrices $(a_{i,j})_{i,j\in\mathbb{N}}$ with $\sum_{i,j\in\mathbb{N}} |a_{i,j}|^2 < \infty$. The algebra $\mathcal{K}$ sits inside $\mathbb{K}$ as a subalgebra.

We define $e_{i,j} \in \mathcal{K}$ as the matrix with 1 in position (i, j) and 0 elsewhere. Then any element of $\mathcal{K}$ can be written as $a = \sum_{i,j=0}^{\infty} a_{i,j}e_{i,j}$.

Lemma 1.28. *The locally convex spaces $\mathcal{K}$, $\mathfrak{s} \otimes_\pi \mathfrak{s}$ and $\mathfrak{s} \oplus \mathfrak{s}$ are isomorphic to $\mathfrak{s}$.*

Proof. The proofs of these facts can be found in [24] Chapter 3 Section 1.1. We give the proofs here for completeness.

First we prove $\mathcal{K} \cong \mathfrak{s}$. Consider the bijection $\phi \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\phi(i, j) = i + \sum_{k=1}^{i+j} k.$$

Note that if $n = \phi(i, j)$, then $i + j \leq n \leq (1 + i + j)^2$.

Given a sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathfrak{s}$, define $a_{ij} = x_n$ for $n = \phi(i, j)$. When $n = \phi(i, j)$, we have $i + j \leq n$ and therefore

$$\sum_{i,j\in\mathbb{N}} (1 + i + j)^k|a_{ij}| \leq \sum_{n\in\mathbb{N}} (1 + n)^k|x_n|.$$

Thus, we have a well-defined continuous map $\psi \colon \mathfrak{s} \to \mathcal{K}$.

Given $(a_{ij})_{i,j\in\mathbb{N}} \in \mathcal{K}$, define $x_n = a_{ij}$ for $(i, j) = \phi^{-1}(n)$. When $n = \phi(i, j)$, we have $n \leq (1 + i + j)^2$ and therefore

$$\begin{aligned}\sum_{n\in\mathbb{N}} (1 + n)^k|x_n| &\leq \sum_{i,j\in\mathbb{N}} (1 + (1 + i + j)^2)^k|a_{ij}| \\ &\leq 2^k \sum_{i,j\in\mathbb{N}} (1 + i + j)^{2k}|a_{ij}|.\end{aligned}$$

Therefore, the map $\psi^{-1}\colon \mathcal{K} \to \mathfrak{s}$ is well-defined and continuous.

Now, we prove that $\mathcal{K} \cong \mathfrak{s} \otimes_\pi \mathfrak{s}$. Let $x = \{x_i\}_{i\in\mathbb{N}}$ and $y = \{y_i\}_{i\in\mathbb{N}}$ be elements of $\mathfrak{s}$. Define $\eta\colon \mathfrak{s} \otimes \mathfrak{s} \to \mathcal{K}$ by $x \otimes y \mapsto a = (a_{ij})_{i,j\in\mathbb{N}}$ with $a_{ij} = x_i y_j$. Using the inequality $(1+i+j)^k \leq (1+i)^k(1+j)^k$ that holds for all $i, j \in \mathbb{N}$, we have

$$\sum_{i,j\in\mathbb{N}} (1+i+j)^k |x_i y_j| \leq \sum_{i,j\in\mathbb{N}} (1+i)^k(1+j)^k |x_i||y_j|.$$

Thus we have proved $q_k(\eta(x \otimes y)) \leq p_k(x)p_k(y)$ (here q_k and p_k are the seminorms that define the topologies of $\mathcal{K}$ and $\mathfrak{s}$ respectively). Now, given $z \in \mathfrak{s} \otimes \mathfrak{s}$ such that $z = \sum_{t=1}^N x^{(t)} \otimes y^{(t)}$ with $x^{(t)}, y^{(t)} \in \mathfrak{s}$, we have

$$\begin{aligned} q_k(\eta(z)) &\leq \sum_{t=1}^N q_k(\eta(x^{(t)} \otimes y^{(t)})) \\ &\leq \sum_{t=1}^N p_k(x^{(t)})p_k(y^{(t)}) \end{aligned}$$

This implies that $q_k(\eta(z)) \leq (p_k \otimes p_k)(z)$ and thus η is well-defined and continuous. To see that η is injective, let $z = \sum_{t=1}^N x^{(t)} \otimes y^{(t)}$ with $x^{(t)}$ for $1 \leq t \leq N$ linearly independent in $\mathfrak{s}$, and assume $\eta(z) = 0$. Column j of $\eta(z)$ is equal to $\sum_{t=1}^N y_j^{(t)} x^{(t)} = 0$, therefore $y^{(t)} = 0$ for all $1 \leq t \leq N$, and this implies $z = 0$.

Now, we prove that η is open. Let $z = \sum_{t=1}^N x^{(t)} \otimes y^{(t)}$ with $x^{(t)}, y^{(t)} \in \mathfrak{s}$. Then we have $z = \sum_{i,j\in\mathbb{N}} \sum_{t=1}^N x_i^{(t)} y_j^{(t)} e_i \otimes e_j$. Hence,

$$\begin{aligned} (p_k \otimes p_k)(z) &\leq \sum_{i,j\in\mathbb{N}} (p_k \otimes p_k)(\sum_{t=1}^N x_i^{(t)} y_j^{(t)} e_i \otimes e_j) \\ &= \sum_{i,j\in\mathbb{N}} |1+i|^k |1+j|^k |\sum_{t=1}^N x_i^{(t)} y_j^{(t)}| \\ &\leq \sum_{i,j\in\mathbb{N}} |1+i+j|^{2k} |\sum_{t=1}^N x_i^{(t)} y_j^{(t)}| \\ &= q_{2k}(\eta(z)). \end{aligned}$$

Therefore, the topology of $\mathfrak{s} \otimes \mathfrak{s}$ inherited from $\mathcal{K}$ is the projective topology.

In order to finish the proof that η is an isomorphism, we show that $\mathfrak{s} \otimes \mathfrak{s}$ is dense in $\mathcal{K}$. Define the sequences $e_i \in \mathfrak{s}$ as having a 1 in position i and zeros elsewhere. Then

$\eta(e_i \otimes e_j) = e_{ij}$ and therefore M_∞, the space of matrices with finite non-zero entries, is contained in $\mathfrak{s} \otimes \mathfrak{s}$. Since M_∞ is dense in $\mathcal{K}$, $\mathfrak{s} \otimes \mathfrak{s}$ is dense in $\mathcal{K}$. We conclude $\mathcal{K} \cong s \otimes_\pi s$. The isomorphism between $\mathfrak{s} \oplus \mathfrak{s}$ and $\mathfrak{s}$ is given by sending $(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}})$ to $\{z_i\}_{i\in\mathbb{N}}$, where $z_{2k} = x_k$ and $z_{2k+1} = y_{2k+1}$ for $k \in \mathbb{N}$. The proof that this map is well-defined and an isomorphism is similar to the previous proofs.

□

Lemma 1.29. *There is an isomorphism $\theta\colon \mathcal{K} \to \mathcal{K} \otimes_\pi \mathcal{K}$ which is diffotopic to the cannonical inclusion $\iota\colon \mathcal{K} \to \mathcal{K} \otimes_\pi \mathcal{K}$ defined by $a \mapsto e_{00} \otimes a$.*

Proof. See Lemma 2.8 in [19]. □

Before defining the smooth Toeplitz algebra we will define the Toeplitz C*-algebra $\mathcal{T}_{C^*}$. We will use the right shift operator $S \in B(\mathbb{H}) = B(l^2(\mathbb{N}))$ defined by $S(e_n) = e_{n+1}$.

Definition 1.30. Let S be the right shift operator on $B(\mathbb{H})$, then we define the Toeplitz algebra as $\mathcal{T}_{C^*} = C^*(S) \subseteq B(\mathbb{H})$, the C*-subalgebra generated by S.

Remark 1.31. Note that $S^*S = 1$ and $SS^* = 1 - e_{00}$, thus S is an isometry which is not unitary. Alternatively, the Toeplitz algebra can be defined abstractly as the universal unital C*-algebra generated by an isometric element which is not unitary.

Lemma 1.32. *There is an exact sequence of* C*-*algebras*

$$0 \to \mathbb{K} \to \mathcal{T}_{C^*} \to C(S^1) \to 0.$$

Therefore, we have an isomorphism $\mathcal{T}_{C^*} \cong \mathbb{K} \oplus C(S^1)$ as vector spaces.

Now, we define the smooth Toeplitz algebra. The Fourier series gives an isomorphism of locally convex spaces between $C^\infty(S^1)$ and the space $\mathfrak{s}$ of rapidly decreasing Laurent series (see Theorem 51.3 in [23])

$$C^\infty(S^1) \cong \{\sum_{i\in\mathbb{Z}} a_i z^i \mid \sum_{i\in\mathbb{Z}} |1+i|^n |a_i| < \infty,\ \forall n \in \mathbb{N}\},$$

where z corresponds to the function $z\colon S^1 \to \mathbb{C}$, $z(t) = t$.

Definition 1.33. The smooth Toeplitz algebra $\mathcal{T}$ is defined by the direct sum of locally convex vector spaces $\mathcal{T} = \mathcal{K} \oplus C^\infty(S^1)$. To define the multiplication we define $v_k = (0, z^k)$ and just write x for an element $(x, 0)$ with $x \in \mathcal{K}$. We denote the elementary matrices in $\mathcal{K}$ by e_{ij} and set $e_{ij} = 0$ for all $i, j < 0$. The multiplication is defined by the following relations

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad v_k e_{ij} = e_{(i+k),j}, \quad e_{ij}v_k = e_{i,(j-k)},$$

for all $i, j, k, l \in \mathbb{Z}$ and

$$v_k v_{-l} = \begin{cases} v_{k-l}(1 - e_{00} - e_{11} - \dots e_{l-1,l-1}) & , l > 0 \\ v_{k-l} & , l \leq 0, \end{cases}$$

for all $k, l \in \mathbb{Z}$.

We can see that $\mathcal{K}$ is a closed ideal in $\mathcal{T}$. As a matter of fact we have the following extension of locally convex algebras.

Lemma 1.34. *There is a linearly split extension of locally convex algebras*

$$0 \to \mathcal{K} \to \mathcal{T} \to C^\infty(S^1) \to 0.$$

The split sends $z \mapsto v$ and $z^{-1} \mapsto v^*$.

The smooth Toeplitz algebra is generated, as a locally convex algebra, by S and S^*. In fact, it satisfies a universal property in the category of m-algebras.

Lemma 1.35 (Satz 6.1 in [9])**.** *$\mathcal{T}$ is the universal unital m-algebra generated by two elements S and S^* satisfying the relation $S^*S = 1$ whose topology is defined by a family of submultiplicative seminorms $\{p_n\}_{n \in \mathbb{N}}$ with the condition that there are positive constants C_n such that*

$$p_n(S^k) \leq C_n(1 + k^n) \quad \textit{and} \quad p_n(S^{*n}) \leq C_n(1 + k^n). \tag{1.1}$$

The following diffotopy is due to [9]. In the context of C*-algebras a homotopy like this one is used to prove Bott periodicity and to construct the Pimsner-Voiculescu sequence. Because $\mathcal{T}$ and $\mathcal{T} \otimes_\pi \mathcal{T}$ are Frechet, the path ϕ_t defines a diffotopy between ϕ_0 and ϕ_1 (see Remark 1.12).

Lemma 1.36 (Lemma 6.2 in [9]). *There is a unital diffotopy $\phi_t\colon \mathcal{T} \to \mathcal{T} \otimes_\pi \mathcal{T}$ such that*

$$\phi_t(S) = S^2S^* \otimes 1 + f(t)(e \otimes S) + g(t)(Se \otimes 1),$$

where $f, g \in \mathbb{C}[0,1]$ are such that $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ and $g(1) = 0$.

Note that $\phi_0(S) = S \otimes 1$ and $\phi_1(S) = S^2S^* \otimes 1 + e \otimes S$. Lemma 1.35 implies that, in order to define a morphism from $\mathcal{T}$ to $\mathcal{T} \otimes_\pi \mathcal{T}$, we only need to check the relations on S and S^* and the bounds of (1.1).

We finish this section with a result for tensoring algebras with a countable basis over $\mathbb{C}$ equipped with the fine topology and the Toeplitz algebra. This result is used to prove Proposition 3.15.

Lemma 1.37. *The locally convex space $A \otimes_\pi \mathfrak{s}$ is isomorphic to the space F of sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ such that*

$$||x||_{\rho,k} = \sum_{n\in\mathbb{N}} |1+n|^k \rho(x(n))$$

is finite for all $k \in \mathbb{N}$ and any continuous seminorm ρ on A, where the topology on F is defined by the seminorms $||\cdot||_{\rho,k}$.

Proof. There is an inclusion $\phi\colon A \otimes \mathfrak{s} \to F$ defined by $a \otimes \alpha \in A \otimes \mathfrak{s} \mapsto \{x_n = \alpha_n a\} \in F$. Let $z = \sum_{t=1}^N a^{(t)} \otimes \alpha^{(t)}$ be an element of $A \otimes \mathfrak{s}$. We have

$$\begin{aligned} ||\phi(z)||_{\rho,k} &= \sum_{n\in\mathbb{N}} \rho\left(\sum_{t=1}^N a^{(t)}\alpha_n^{(t)}\right) |1+n|^k \\ &\leq \sum_{n\in\mathbb{N}} \sum_{t=1}^N \rho(a^{(t)})|\alpha_n^{(t)}||1+n|^k \\ &= \sum_{t=1}^N \rho(a^{(t)})p_k(\alpha^{(t)}). \end{aligned}$$

This implies $||\phi(z)||_{\rho,k} \leq (\rho \otimes p_k)(z)$. We can write $z = \sum_{n\in\mathbb{N}} \sum_{t=1}^N a^{(t)}\alpha_n^{(t)} \otimes e_n$ and therefore

$$\begin{aligned} (\rho \otimes p_k)(z) &\leq \sum_{n\in\mathbb{N}} (\rho \otimes p_k)\left(\sum_{t=1}^N a^{(t)}\alpha_n^{(t)} \otimes e_n\right) \\ &= \sum_{n\in\mathbb{N}} \rho\left(\sum_{t=1}^N a^{(t)}\alpha_n^{(t)}\right) |1+n|^k \\ &= ||\phi(z)||_{\rho,k}. \end{aligned}$$

This implies that $||\cdot||_{\rho,k} = \rho\otimes p_k$ in the image of $A\otimes\mathfrak{s}$. Since all finite sequences in A are in $A\otimes\mathfrak{s}$, $A\otimes\mathfrak{s}$ is dense in F. Since F is a complete space, we conclude $A\otimes_\pi\mathfrak{s} = F$. $\square$

Lemma 1.38. *Let $\mathfrak{s}$ be the locally convex space of rapidly decreasing sequences and A an algebra with a countable basis over $\mathbb{C}$ equipped with the fine topology. Then, we have*

$$A\otimes_\pi\mathfrak{s} = A\otimes\mathfrak{s},$$

as locally convex spaces. This implies that

$$A\otimes_\pi\mathcal{T} = A\otimes\mathcal{T} \quad \text{and} \quad A\otimes_\pi(\mathcal{T}\otimes_\pi\mathcal{T}) = A\otimes(\mathcal{T}\otimes_\pi\mathcal{T}),$$

as locally convex algebras.

Proof. We prove that the space F from Lemma 1.37 is equal to the algebraic tensor product $A\otimes\mathfrak{s}$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a countable basis of A. Given $\{x_n\}_{n\in\mathbb{N}}$ a sequence of elements in A with $\rho_k(x)$ finite for all $k\in\mathbb{N}$ we have, for n fixed

$$x_n = \sum_{i\in\mathbb{N}}\lambda_n^{(i)}v_i,$$

where $\lambda_n^{(i)}\neq 0$ for finitely many $i\in\mathbb{N}$.

First, we prove that $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$ is finite dimensional. Suppose this is not the case. We construct subsequences $\{x_{n_i}\}$ and $\{v_{m_i}\}$ such that $\lambda_{n_i}^{(m_i)}\neq 0$. Choose n_1 such that $x_{n_1}\neq 0$ and m_1 such that $\lambda_{n_1}^{(m_1)}\neq 0$. Suppose $\{x_{n_1},\ldots,x_{n_k}\}$ and $\{v_{m_1},\ldots,v_{m_k}\}$ have been chosen. Notice that $\operatorname{span}\{x_i\}_{i>n_k}$ is infinite dimensional, and therefore it is not contained in $\operatorname{span}\{v_i\}_{1\le i\le m_k}$. Choose $n_{k+1}>n_k$ such that $x_{n_{k+1}}\notin\operatorname{span}\{v_i\}_{1\le i\le m_k}$. We can choose $m_{k+1}>m_k$ such that $\lambda_{n_{k+1}}^{(m_{k+1})}\neq 0$.

Now we define a seminorm in A,

$$\rho(\sum_{i\in\mathbb{N}}c_iv_i) = \sum_{i\in\mathbb{N}}|c_i|\alpha_i,$$

with $\alpha_i = 0$ for $i\notin\{n_k\}_{k\in\mathbb{N}}$ and $\alpha_{n_k}\geq|\lambda_{n_k}^{(m_k)}|^{-1}$. Thus we have $\rho(x_{n_k})\geq 1$ and

$$\rho_0(x) = \sum_{i\in\mathbb{N}}\rho(x_i)\geq\sum_{i\in\mathbb{N}}\rho(x_{n_i})$$

diverges. We conclude that $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$ is finite dimensional.

Let $N \in \mathbb{N}$ be such that $\text{span}\{x_n\}_{n\in\mathbb{N}} \subseteq \text{span}\{v_1, \ldots, v_N\}$. That is

$$x_n = \sum_{i=0}^{N} \lambda_n^{(i)} v_i$$

Then

$$\begin{aligned} x &= \lim_{M\to\infty} \sum_{n=0}^{M} x_n \otimes e_n \\ &= \lim_{M\to\infty} \sum_{n=0}^{M} \sum_{i=0}^{N} \lambda_n^{(i)} v_i \otimes e_n \\ &= \lim_{M\to\infty} \sum_{i=0}^{N} v_i \otimes \sum_{n=0}^{M} \lambda_n^{(i)} e_n. \end{aligned}$$

Consider the seminorm $p_j(\sum c_i v_i) = |c_j|$. Then, since $x \in A \otimes_\pi \mathfrak{s}$,

$$\sum_{n\in\mathbb{N}} |1+n|^k p_i(x_n) = \sum_{n\in\mathbb{N}} |1+n|^k |\lambda_n^{(i)}| < \infty$$

for all $k \in \mathbb{N}$. Thus, for a fixed i, the sequences $\{\lambda_n^{(i)}\}$ are rapidly decreasing on n. Therefore $\sum_{n\in\mathbb{N}} \lambda_i^{(n)} e_n \in \mathfrak{s}$, and consequently $x = \sum_{i=1}^{N} v_i \otimes s_i \in A \otimes \mathfrak{s}$.

The equalities $A \otimes_\pi \mathcal{T} = A \otimes \mathcal{T}$ and $A \otimes_\pi (\mathcal{T} \otimes_\pi \mathcal{T}) = A \otimes (\mathcal{T} \otimes_\pi \mathcal{T})$ follow because, as locally convex vector spaces, we have $\mathcal{T} \cong \mathfrak{s}$ and $\mathcal{T} \otimes_\pi \mathcal{T} \cong \mathfrak{s}$. $\square$

Chapter 2

Bivariant K-theory

In this chapter, following [11], we give the definition of the supension stable categoryΣHo, state its main properties and then describe the relation between kk^{alg} and ΣHo. For a complete treatise of these constructions in the context of bornological algebras consult [11]. We also study weak Morita equivalences and quasi-homomorphisms. We finish the chapter summarizing the results that have been obtained for computing the invariants of $\mathbb{Z}$-graded algebras.

2.1 The suspension stable category

First, we construct the category ΣHo (see Section 6.3 in [11]). The objects of ΣHo are pairs (A, n) where A is a locally convex algebra and $n \in \mathbb{Z}$. Given two objects (A, n) and (B, m) of ΣHo, the set of morphisms is defined as

$$\Sigma\text{Ho}((A,n),(B,m)) = \varinjlim_{k\in\mathbb{N}} \langle J^{n+k}A, S^{m+k}B\rangle$$

where the inductive limit is taken over $k \in \mathbb{N}$ with $n+k, m+k \geq 0$. The inductive system is defined by sending the diffotopy class of $\alpha\colon J^{n+k}A \to S^{m+k}B$ to the morphism $\alpha'\colon J^{n+k+1}A \to S^{m+k+1}B$ defined up to diffotopy as the classifying map for the second row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^{n+k+1}A & \longrightarrow & T(J^{n+k}A) & \longrightarrow & J^{n+k}A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & S^{m+k+1}B & \longrightarrow & C(S^{m+k}B) & \longrightarrow & S^{m+k}B & \longrightarrow & 0 \end{array}$$

(see definition 1.24). We note that $\Sigma\mathrm{Ho}((A,n),(B,m))$ has the structure of an abelian group because of Lemma 1.15

The composition of morphisms in $\Sigma\mathrm{Ho}$ is defined as follows. Given morphisms in $\Sigma\mathrm{Ho}$ $(A,n) \to (B,m)$ and $(B,m) \to (C,p)$ with representatives

$$f\colon J^{n+k}A \to S^{m+k}B \quad \text{and} \quad g\colon J^{m+l}B \to S^{p+l}C$$

we take the composition $(A,n) \to (C,p)$ to be defined by the composition

$$\begin{array}{ccccccc} J^{m+l}J^{n+k}A & \xrightarrow{J^{m+l}(f)} & J^{m+l}S^{m+k}B & \longrightarrow & S^{m+k}J^{m+l}B & \xrightarrow{S^{m+k}(g)} & S^{m+k}S^{p+l}C \\ \uparrow{=} & & & & & & \uparrow{=} \\ J^{n+(m+k+l)}A & & & & & & S^{p+(m+k+l)}C \end{array}$$

The morphism in the center is $(-1)^{(m+l)(m+k)}\kappa_B^{(m+k),(m+l)}$, where $\kappa_B^{i,j}$ is the projection defined in 1.26.

Remark 2.1. The proof that this definition is independent of the representative chosen in the direct limit and that it is associative requires will be omitted. The reader can find this proof in Section 6.3 of [11].

The category $\Sigma\mathrm{Ho}$ has a suspension functor

$$\Sigma\colon \Sigma\mathrm{Ho} \to \Sigma\mathrm{Ho}$$

defined by $\Sigma(A,n) = (A,n+1)$ and sending a morphism $f\colon (A,n) \to (B,m)$ to a morphism $\Sigma f\colon (A,n+1) \to (B,m+1)$ with the same representative in the direct limit. There is an inverse functor $\Sigma^{-1}\colon \Sigma\mathrm{Ho} \to \Sigma\mathrm{Ho}$ with $\Sigma^{-1}(A,n) = (A,n-1)$ and defined for morphisms in an analog way as Σ. Thus Σ is an automorphism.

There are also two functors $S, J\colon \Sigma\mathrm{Ho} \to \Sigma\mathrm{Ho}$. S is defined by $S(A,n) = (SA,n)$ and it sends a morhism $f\colon (A,n) \to (B,m)$ with representative $f\colon J^{n+k}A \to S^{m+k}B$ to the morphism in $\Sigma\mathrm{Ho}$ defined by the composition

$$J^{n+k}SA \xrightarrow{(-1)^{n+k}\kappa_A^{1,n+k}} SJ^{n+k}A \xrightarrow{S(f)} SS^{m+k}B.$$

J is defined by $J(A,n) = (JA,n)$ and it sends a morphism $f\colon (A,n) \to (B,m)$ with representative $f\colon J^{n+k}A \to S^{m+k}B$ to the morphism in ΣHo defined by the composition

$$J^{n+k}JA \xrightarrow{\quad J(f) \quad} JS^{m+k}B \xrightarrow{(-1)^{m+k}\kappa_B^{m+k,1}} S^{m+k}JB.$$

Lemma 2.2. *The functors S and J are isomorphic to the suspension Σ in ΣHo.*

Proof. See Lemmas 6.29 and 6.30 in [11]. □

Now we state the universal property of the suspension-stable homotopy category. Note that there is a functor from the category of locally convex algebras to ΣHo that sends a locally convex algebra A to $(A,0)$ and a morphism $f\colon A \to B$ to its representative in $\Sigma\mathrm{Ho}((A,0),(B,0))$. We denote this functor again by ΣHo. Similarly, we have functors $\Sigma\mathrm{Ho}_n$ from the category of locally convex algebras to abelian groups defined by sendig A to (A,n) and sending $f\colon A \to B$ to the class of $\alpha\colon J^nA \to S^nB$, the classifying map of

$$0 \to S^nB \to CS^{n-1}B \to \cdots \to CB \to B \to 0.$$

The functors $\{\Sigma\mathrm{Ho}_n\}_{n\in\mathbb{Z}}$ define a homology theory for locally convex algebras as defined below.

Definition 2.3. A functor F from the category of locally convex algebras to an abelian category is called

1. diffotopy invariant if $F(f) = F(g)$ whenever f and g are diffotopic,
2. half exact for linearly split extensions if

$$F(A) \to F(B) \to F(C)$$

   is exact whenever

$$0 \to A \to B \to C \to 0$$

   is a linearly split extension.

Definition 2.4. A homology theory for locally convex algebras is a sequence of covariant functors $\{F_n\}_{n\in\mathbb{Z}}$ from the category of locally convex algebras to an abelian category together with natural isomorphisms $F_n(SA) \cong F_{n+1}(A)$ for all $n \in \mathbb{Z}$, such that

1. the functors F_n are diffotopy invariant;
2. the functors F_n are half exact for linearly split extensions.

Proposition 2.5 (Proposition 6.72 in [11]). *If $\{F_n\}_{n\in\mathbb{Z}}$ is a homology theory for bornological algebras, then $\bar{F}(A,n) := F_n(A)$ defines a homological functor $\bar{F}\colon \Sigma Ho \to Ab$. Conversely, any such homological functor $\bar{F}$ arises from a unique homology theory for bornological algebras in this fashion.*

2.2 Definition of kk^{alg}

We define kk^{alg} and describe its relation to ΣHo. The functor $\Sigma\mathrm{Ho}\colon \mathfrak{lca} \to \Sigma\mathrm{Ho}$ still lacks some properties like Bott periodicity. To obtain this property we have to stabilize our algebras. The stabilization that Cuntz considered in [10] is given by $\mathcal{K}$, the algebra of smooth compact operators. This is the smallest algebra that we can consider to obtain Bott periodicity. The problem with this stabilization is that the coefficient ring $kk^{\mathrm{alg}}(\mathbb{C},\mathbb{C})$ is difficult to compute. Other stabilizations such as stabilization by the Schatten ideals $\mathcal{L}^p$ have been considered in [12]. Considering this kinds of ideals the coefficient ring can be computed to be $kk_*^{\mathcal{L}} = \mathbb{Z}[u,u^{-1}]$ with u in degree 2.

The following is the definition of kk^{alg} as in [10].

Definition 2.6. Let A and B be locally convex algebras. We define

$$kk^{\mathrm{alg}}(A,B) = \varinjlim_{k\in\mathbb{N}} \langle J^k A, \mathcal{K}\otimes_\pi S^k B\rangle$$

and for $n \in \mathbb{N}$

$$kk_n^{\mathrm{alg}}(A,B) = kk^{\mathrm{alg}}(J^n A, B), \quad kk_{-n}^{\mathrm{alg}}(A,B) = kk^{\mathrm{alg}}(A, S^n B).$$

Note that we have defined $kk^{\mathrm{alg}}(A,B) = \Sigma\mathrm{Ho}(A, \mathcal{K}\otimes_\pi B)$. The following Lemma will tell us that this definition is equivalent to $kk^{\mathrm{alg}}(A,B) = \Sigma\mathrm{Ho}(\mathcal{K}\otimes_\pi A, \mathcal{K}\otimes_\pi B)$.

Lemma 2.7 (Lemma 7.21 in [11]). *Composition with the stabilization $A \to \mathcal{K}\otimes_\pi A$ induces a natural isomorphism*

$$\Sigma Ho(\mathcal{K}\otimes_\pi A, \mathcal{K}\otimes_\pi B) \cong \Sigma Ho(A, \mathcal{K}\otimes_\pi B)$$

We can therefore view kk^{alg} as a quotient category of ΣHo.

The associative product of kk^{alg} follows from the associative product of ΣHo.

Lemma 2.8. *There is an associative product*

$$kk_n^{\mathrm{alg}}(A,B) \times kk_m^{\mathrm{alg}}(B,C) \to kk_{n+m}^{\mathrm{alg}}(A,C)$$

Proof. Follows from Lemma 6.32 in [11]. □

In view of this associative product we can regard locally convex algebras as objects of a category $\mathfrak{KK}^{\mathrm{alg}}$ with morphisms between A and B given by elements of $kk_*^{\mathrm{alg}}(A,B)$. Any morphism $\phi\colon A \to B$ of locally convex algebras induces an element $kk(\phi) \in kk^{\mathrm{alg}}(A,B)$ which is associated with the diffotopy class of $i \circ \phi\colon A \to B \to \mathcal{K} \otimes_\pi B$, where i is the inclusion of B into the first corner, i.e. $i(b) = e_{00} \otimes b$.

Lemma 2.9 (see Theorem 2.3.1 in [12])**.** *If $\phi\colon A \to B$ and $\psi\colon B \to C$ are morphisms of locally convex algebras then*

$$kk(\phi)kk(\psi) = kk(\psi \circ \phi).$$

It can also be seen that the identity of A induces $kk(\mathrm{id}_A) = 1_A \in \mathfrak{KK}^{\mathrm{alg}}(A,A)$, the identity of A in the category $\mathfrak{KK}^{\mathrm{alg}}$. Therefore there is a functor

$$kk_*^{\mathrm{alg}}\colon \mathfrak{lca} \to \mathfrak{KK}^{\mathrm{alg}}.$$

Remark 2.10. In what follows, given two morphisms $\phi : A \to B$ and $\phi : B \to C$in $\mathfrak{lca}$, we shall denote the product

$$kk(\phi)kk(\psi)$$

as a composition in $\mathfrak{KK}^{\mathrm{alg}}$

$$kk(\psi) \circ kk(\phi).$$

Extensions of A by B of length n that are linearly split define elements in $kk_{-n}^{\mathrm{alg}}(A,B)$. This is because to any linearly split extension

$$E\colon 0 \to B \to E_n \to \cdots \to E_1 \to A \tag{2.1}$$

there corresponds the diffotopy class of a morphism $J^nA \to B$. We will call this element $kk(E) \in kk_{-n}^{\mathrm{alg}}(A,B)$.

Lemma 2.11. *Given two extensions of lenghts n and m*

$$E_1\colon \qquad 0 \to B \to E_n \to \cdots \to E_1 \to A$$
$$E_2\colon \qquad 0 \to C \to D_m \to \cdots \to D_1 \to B$$

the product $kk(E_2)kk(E_1) = kk(E)$ where E is the Yoneda product of E_1 and E_2

$$E\colon \qquad 0 \to C \to D_m \to \cdots \to D_1 \to E_n \to \cdots \to E_1 \to A.$$

We now turn to study the properties of kk^{alg}. These are diffotopy invariance, half exactness for linearly split extensions and stability with respect to $\mathcal{K}$. We will see that kk^{alg} is universal with respect to homological functors into some abelian category that satisfy these properties.

Definition 2.12. A functor F from the category of locally convex algebras to an abelian category is called $\mathcal{K}$-stable if the natural inclusion $i\colon A \to \mathcal{K} \otimes_\pi A$, sending a to $e \otimes a$ induces an isomorphism $F(i)\colon F(A) \to F(\mathcal{K} \otimes_\pi A)$.

Proposition 2.13. *The functor $kk^{\mathrm{alg}}\colon \mathfrak{lca} \to kk^{\mathrm{alg}}$ is diffotopy invariant, half exact for linearly split extensions and is $\mathcal{K}$-stable.*

As a matter of fact, the functor kk^{alg} is universal with respect to these properties.

Theorem 2.14 (Theorem 7.26 in [11])**.** *If F is a covariant functor from the category of bornological algebras to an abelian category $\mathfrak{C}$ that is diffotopy invariant, half exact for linearly split extensions and $\mathcal{K}$-stable then $F = \bar{F} \circ kk^{\mathrm{alg}}$ for a unique homological functor $\bar{F}\colon kk^{\mathrm{alg}} \to \mathfrak{C}$.*

2.3 Stabilization by Schatten ideals

In [12], Cuntz and Thom define a related bivariant K-theory in the category $\mathfrak{lca}$. We recall the definition for the case of the Schatten ideals. Let $\mathbb{H}$ denote an infinite dimensional separable Hilbert Space.

Definition 2.15. The Schatten ideals $\mathcal{L}_p \subseteq B(\mathbb{H})$, for $p \geq 1$, are defined by

$$\mathcal{L}_p = \{x \in B(\mathbb{H})| \ Tr|x|^p < \infty\}.$$

Equivalently, $\mathcal{L}_p$ consists of the space of bounded operators such that the sequence of its singular values $\{\mu_n\}$ is in $l^p(\mathbb{N})$.

Definition 2.16. Let A and B be locally convex algebras and $p \geq 1$. We define

$$kk_n^{\mathcal{L}_p}(A,B) = kk^{\mathrm{alg}}(A, B \otimes_\pi \mathcal{L}_p).$$

The groups $kk^{\mathcal{L}_p}(A,B)$, for all $p \geq 1$, are isomorphic (Corollary 2.3.5 of [12]).

This bivariant K-theory is related to algebraic K-theory when $p > 1$.

Theorem 2.17 (Theorem 6.2.1 in [12])**.** *For every locally convex algebra A and $p > 1$ we have*

$$kk_0^{\mathcal{L}_p}(\mathbb{C}, A) = K_0(A \otimes_\pi \mathcal{L}_p).$$

Corollary 2.18 (Corollary 6.2.3 in [12])**.** *The coefficient ring $kk_*^{\mathcal{L}_p}(\mathbb{C},\mathbb{C})$ is isomorphic to $\mathbb{Z}[u,u^{-1}]$ with $deg(u) = 2$.*

This implies that $kk_0^{\mathcal{L}_p}(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ and $kk_1^{\mathcal{L}_p}(\mathbb{C},\mathbb{C}) = 0$.

2.4 Bott periodicity and Triangulated structure of $\mathfrak{KK}^{\mathrm{alg}}$

The suspension of locally convex algebras determines a functor $S\colon \mathfrak{KK}^{\mathrm{alg}} \to \mathfrak{KK}^{\mathrm{alg}}$ with $S(A) = SA$.

Theorem 2.19. *[Bott periodicity] There is a natural equivalence between S^2 and the identity functor, hence $\mathfrak{KK}_{2n}^{\mathrm{alg}}(A,B) \cong \mathfrak{KK}_0^{\mathrm{alg}}(A,B)$ and $\mathfrak{KK}_{2n+1}^{\mathrm{alg}}(A,B) \cong \mathfrak{KK}_1^{\mathrm{alg}}(A,B)$.*

Proof. See Corollary 7.25 in [11] and the discussion that follows. □

By Theorem 2.19, S is an automorphism and $S^{-1} \cong S$. We recall the triangulated structure of $(\mathfrak{KK}, S)$.

Let $f\colon A \to B$ be a morphism in $\mathfrak{lca}$. The mapping cone of f is defined as the locally convex algebra

$$C(f) = \{(x, g) \in A \oplus CB | f(x) = g(0)\}.$$

The triangle

$$SB \xrightarrow{kk(\iota)} C(f) \xrightarrow{kk(\pi)} A \xrightarrow{kk(f)} B$$

in $(\mathfrak{KK}^{\mathrm{alg}}, S)$, where $\pi\colon C(f) \to A$ is the projection into the first component and $\iota\colon SB \to C(f)$ is the inclusion into the first component, is called a mapping cone triangle.

Let $E\colon 0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be a linearly split extension in $\mathfrak{lca}$. This induces an element $kk(E) \in kk_1^{\mathrm{alg}}(C, A)$ that corresponds to the classifying map $JC \to A$ of the extension and hence an element $kk(E) \in kk^{\mathrm{alg}}(SC, A)$. The triangle

$$SC \xrightarrow{kk(E)} A \xrightarrow{kk(f)} B \xrightarrow{kk(g)} C$$

in $(\mathfrak{KK}^{\mathrm{alg}}, S)$ is called an extension triangle.

Proposition 2.20. *The category $\mathfrak{KK}^{\mathrm{alg}}$ with suspension automorphism $S\colon \mathfrak{KK}^{\mathrm{alg}} \to \mathfrak{KK}^{\mathrm{alg}}$ and with triangles isomorphic to mapping cone triangles as exact triangles is a triangulated category. Furthermore, extension triangles are exact.*

Proof. See Propositions 7.22 and 7.23 in [11]. □

2.5 Weak Morita equivalence

In the context of separable C*-algebras, two algebras A and B are strong Morita equivalent if and only if $\mathbb{K} \otimes A \cong \mathbb{K} \otimes B$ (they are stably isomorphic). Therefore a strong Morita equivalence of separable C*-algebras induces an equivalence in Kasparov's bivariant K-theory, KK. In the case of locally convex algebras we define weak Morita equivalence, which still give us an isomorphism between two objects in $\mathfrak{KK}^{\mathrm{alg}}$. This is a weak version of Morita equivalence.

A Morita context gives us the data needed to define maps $A \to \mathcal{K} \otimes_\pi B$.

Definition 2.21. Let A and B be locally convex algebras. A Morita context from A to B consists of a locally convex algebra E that contains A and B as subalgebras and two sequences $(\xi_i)_{i\in\mathbb{N}}$ and $(\eta_j)_{j\in\mathbb{N}}$ of elements of E that satisfy

1. $\eta_j A \xi_i \subset B$ for all i, j.
2. The sequence $(\eta_j a \xi_i)$ is rapidly decreasing for each $a \in A$. That is, for each continuous seminorm α in B, $\alpha(\eta_j a \xi_i)$ is rapidly decreasing in i, j.
3. For all $a \in A$, $(\sum \xi_i \eta_i) a = a$.

A Morita context $((\xi_i), (\eta_j))$ from A to B determines a homomorphism $A \to \mathcal{K} \otimes_\pi B$ defined by $a \mapsto \sum_{i,j\in\mathbb{N}} e_{ij} \otimes \eta_j a \xi_i$. Thus it determines an element $kk((\xi_i), (\eta_j))$ of $kk_0^{\mathrm{alg}}(A, B)$, which is an element of $\mathfrak{KK}^{\mathrm{alg}}(A, B)$.

In the next proposition, we give conditions for a Morita context to determine an equivalence in $\mathfrak{KK}^{\mathrm{alg}}$.

Proposition 2.22. *Let $((\xi_i), (\eta_j))$ be a Morita context from A to B in E. If $((\xi'_l), (\eta'_k))$ is a Morita context from B to A in the same locally convex algebra and if $A\xi_i\xi'_l \subset A$ and $\eta'_k\eta_j A \subset A$ for all i, j, k, l; then*

$$kk((\xi'_l), (\eta'_k)) \circ kk((\xi_i), (\eta_j)) = 1_A.$$

Therefore, if we also have $B\xi'_l\xi_i \subset B$ and $\eta_k\eta'_j B \subset B$ for all i, j, k, l, then $kk((\xi_i), (\eta_j))$ is invertible in $\mathfrak{KK}^{\mathrm{alg}}$.

Proof. See Lemma 7.2 in [10]. □

2.6 Quasi-homomorphisms

The definition of a quasi-homomophism goes back to [5]. In that article, given a pair of separable C*-algebras A and B, $KK(A, B)$ is characterized as the set of homotopy classes of quasi-homomorphisms between A and $B \otimes \mathbb{K}$. With this characterization the product in KK can be obtained in an easy way [8].

We will define quasi-homomorphisms in the context of locally convex algebras. In this context, quasi-homomorphisms will still define elements of kk^{alg} and because of their properties they will be useful to prove our results. As a matter of fact we can make the definition for an arbitrary split-exact functor $E\colon \mathfrak{lca} \to \mathfrak{C}$ where $\mathfrak{C}$ is an additive category. In this section we follow Section 4 of [15], Section 3.3.1 in [11] and Section 3 in [12].

Definition 2.23. Let A, B and D be locally convex algebras with B a closed subalgebra of D. A quasi-homomorphism from A to B in D is a pair of homomorphisms $(\alpha, \bar{\alpha})$ from A to D such that $\alpha(x) - \bar{\alpha}(x) \in B$, $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ for all $x \in A$. We denote such quasi-homomorphism by $(\alpha, \bar{\alpha})\colon A \rightrightarrows D \triangleright B$.

Remark 2.24. Some definitions of quasi-homomorphisms require B to be an ideal in D. However, we will need Definition 2.27. Note that if B is an ideal then the conditions $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ are satisfied for all $x \in A$. As a matter of fact we only need to check these conditions in a set of algebraic generators of A.

Lemma 2.25. *Let $G \subset A$ be a subset of algebraic generators of A. If $\alpha(x) - \bar{\alpha}(x) \in B$, $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ for all $x \in G$, then the same conditions are also satisfied for all $x \in A$.*

Proof. If $x, y \in G$, then it is easy to see that $x + y$ also satisfies the conditions. Next we show that xy also satisfies the conditions. Note that

$$(\alpha(x) - \bar{\alpha}(x))(\alpha(y) - \bar{\alpha}(y)) = \alpha(x)(\alpha(y) - \bar{\alpha}(y)) - \bar{\alpha}(x)(\alpha(y) - \bar{\alpha}(y)) \in B.$$

Since $\alpha(x)(\alpha(y) - \bar{\alpha}(y)) \in B$, then $\bar{\alpha}(x)(\alpha(y) - \bar{\alpha}(y)) \in B$ from this we have

$$\alpha(xy) - \bar{\alpha}(xy) = (\alpha(x) - \bar{\alpha}(x))\alpha(y) + \bar{\alpha}(x)(\alpha(y) - \bar{\alpha}(y)) \in B.$$

Similarly we can show $\alpha(xy)B, B\alpha(xy) \subset B$. □

Next we see how a quasi-homomorphism $(\alpha, \alpha')\colon A \rightrightarrows D \triangleright B$ determines an element $kk(\alpha, \bar{\alpha}) \in kk^{\mathrm{alg}}(A, B)$. As a matter of fact, we work with split exact functors from $\mathfrak{lca}$ to an additive category $\mathfrak{C}$. An extension $0 \longrightarrow A \longrightarrow B \stackrel{\pi}{\longrightarrow} C \longrightarrow 0$ in $\mathfrak{lca}$ is split if there is a morphism of locally convex algebras $s\colon C \to B$ such that $\pi s = \mathrm{id}_C$.

Definition 2.26. Let $\mathfrak{C}$ be an additive category. A sequence $A \to B \to C$ in $\mathfrak{C}$ is split exact if it is isomorphic to the sequence $A \to A \oplus C \to C$ with the natural inclusion and projection. A functor $E\colon \mathfrak{lca} \to \mathfrak{C}$ is called split exact if it sends split extensions in $\mathfrak{lca}$ to split exact sequences in $\mathfrak{C}$.

Lemma 2.27. *[Section 3.2 in* [12]*] Let E be a split exact functor from $\mathfrak{lca}$ to an additive category $\mathfrak{C}$. Then a quasi-homomorphism $(\alpha, \alpha')\colon A \rightrightarrows D \triangleright B$ determines a morphism $E(\alpha, \bar{\alpha})\colon E(A) \to E(B)$ in $\mathfrak{C}$.*

Proof. Let D' be the closed subalgebra of $A \oplus D$ generated by all elements $(a, \alpha(a))$ and $(0, b)$ with $a \in A$ and $b \in B$. Then we have an exact sequence

$$0 \to B \to D' \to A \to 0$$

with the inclusion $B \subseteq D'$ given by $b \mapsto (0, b)$ and the projection $\pi\colon D' \to A$ defined by $\pi(a, x) = a$. This extension has two splits $\alpha', \bar{\alpha}'\colon A \to D'$ defined by $\alpha'(a) = (a, \alpha(a))$ and $\bar{\alpha}'(a) = (a, \bar{\alpha}(a))$. Because of the split-exactness of E, $E(B) \to E(D')$ is a kernel of $E(D') \to E(A)$. Therefore, the morphism $E(\alpha') - E(\bar{\alpha}')\colon E(A) \to E(D')$ defines a morphism $E(\alpha, \bar{\alpha})\colon E(A) \to E(B)$. ☐

The following proposition summarizes some properties of quasi-homomorphisms.

Proposition 2.28. *Let E be a split exact functor from $\mathfrak{lca}$ to an additive category $\mathfrak{C}$ and $(\alpha, \alpha')\colon A \rightrightarrows D \triangleright B$ be a quasi-homomorphism from A to B in D.*

1. $E(\alpha, \bar{\alpha}) = -E(\bar{\alpha}, \alpha)$
2. *For any morphism $\phi\colon A' \to A$, $(\alpha \circ \phi, \bar{\alpha} \circ \phi)\colon A' \to B$ is a quasi-homomorphism from A' to B in D and*

$$E(\alpha \circ \phi, \bar{\alpha} \circ \phi) = E(\alpha, \bar{\alpha}) \circ E(\phi)$$

3. *If $\psi\colon D \to F$ is a morphism such that $\psi|_B\colon B \to C \subset F$ and the morphisms $\psi \circ \alpha, \psi \circ \bar{\alpha}\colon A \to F$ define a quasi-homomorphism from A to C in F, then*

$$E(\psi \circ \alpha, \psi \circ \bar{\alpha}) = E(\psi|_B) \circ E(\alpha, \bar{\alpha})$$

4. *Let $\phi = \alpha - \bar{\alpha}$. If $\phi(x)\bar{\alpha}(y) = \bar{\alpha}(y)\phi(x) = 0$ for all $x, y \in A$, then ϕ is a homomorphism and $E(\alpha, \bar{\alpha}) = E(\phi)$*

5. *Let α and $\bar{\alpha}$ be homomorphisms from A to $D[0,1]$ such that $\alpha(x) - \bar{\alpha}(x) \in B[0,1]$, $\alpha(x)B[0,1] \subset B[0,1]$ and $B[0,1]\alpha(x) \subset B[0,1]$ for all $x \in A$. If E is diffotopy invariant, then $E(\alpha_1, \bar{\alpha}_1) = E(\alpha_0, \bar{\alpha}_0)$ (where $\alpha_t = \mathrm{ev}_t \circ \alpha$).*

Proof. The proofs of (1)-(4) can be found in Proposition 21 of [16]. We give them here for completion. (1) follows from the fact that $E(\bar{\alpha}) - E(\alpha) = -(E(\alpha) - E(\bar{\alpha}))$.

For (2) it is easy to see that the morphisms $\alpha \circ \phi, \bar{\alpha} \circ \phi \colon A' \to D$ define a quasi-homomorphism from A' to B in D. If we define $D' \subseteq A \oplus D$ as in the proof of Lemma 2.27 and $D'' \subseteq A' \oplus D$ as the subalgebra generated by the elements $(a', (\alpha \circ \phi)(a'))$ and $(0, b)$ for all $a' \in A'$ and $b \in B$, then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i'} & D' & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\psi}\uparrow & & {\scriptstyle\phi}\uparrow & & \\ 0 & \longrightarrow & B & \xrightarrow{i''} & D'' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

where $\psi \colon D' \to D''$ is the restriction of $\phi \oplus \mathrm{id}_D \colon A' \oplus D \to A \oplus D$. The second row has the splits $\lambda(a') = (a', (\alpha \circ \phi)(a))$ and $\bar{\lambda}(a') = (a', (\alpha' \circ \phi)(a'))$. Now we have

$$\begin{aligned} E(\psi) \circ (E(\lambda) - E(\bar{\lambda})) &= (E(\alpha') - E(\bar{\alpha}')) \circ E(\phi) \\ E(\psi) \circ E(i'') \circ E(\alpha \circ \phi, \bar{\alpha} \circ \phi) &= E(i') \circ E(\alpha, \bar{\alpha}) \circ E(\phi) \\ E(i') \circ E(\alpha \circ \phi, \bar{\alpha} \circ \phi) &= E(i') \circ E(\alpha, \bar{\alpha}) \circ E(\phi) \end{aligned}$$

and from the injectivity of $E(i')$ we deduce $E(\alpha \circ \phi, \bar{\alpha} \circ \phi) = E(\alpha, \bar{\alpha}) \circ E(\phi)$.

We have a similar situation in (3). Define F' the subalgebra of $A \oplus F$ associated to the quasi-homomorphisms $(\psi \circ \alpha, \psi \circ \bar{\alpha})$ defined as in the proof of Lemma 2.27. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i'} & D' & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi_B} & & \downarrow{\scriptstyle\eta} & & \| & & \\ 0 & \longrightarrow & C & \xrightarrow{i''} & F' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Where η is the restriction of $\mathrm{id}_A \oplus \psi \colon A \oplus D \to A \oplus F$ and the second row has splits $\beta(a) = (a, (\psi \circ \alpha)(a))$ and $\bar{\beta}(a) = (a, (\psi \circ \bar{\alpha})(a))$. Now we have

$$
\begin{aligned}
E(\beta) - E(\bar{\beta}) &= E(\eta \circ \alpha') - E(\eta \circ \bar{\alpha}') \\
E(i'') \circ E(\psi \circ \alpha, \psi \circ \bar{\alpha}) &= E(\eta) \circ (E(\alpha') - E(\bar{\alpha}')) \\
&= E(\eta) \circ E(i') \circ E(\alpha, \bar{\alpha}) \\
&= E(i'') \circ E(\psi_B) \circ E(\alpha, \bar{\alpha}).
\end{aligned}
$$

Again, from the injectivity of $E(i'')$, we conclude $E(\psi \circ \alpha, \psi \circ \bar{\alpha}) = E(\psi_B) \circ E(\alpha, \bar{\alpha})$.

(4) follows from the fact that E is split exact and therefore it respects direct sums. In the case that $\alpha - \bar{\alpha}$ is a morphism, then we have $E(\alpha - \bar{\alpha}) = E(\alpha) - E(\bar{\alpha})$. Considering $\alpha - \bar{\alpha} \colon A \to B$, we obtain $E(\alpha - \bar{\alpha}) = E(\alpha, \bar{\alpha})$.

To prove (5), we consider the evaluation maps $\mathrm{ev}_t \colon D[0,1] \to D$. They restrict to the evaluation maps $\mathrm{ev}_t \colon B[0,1] \to B$. To apply (3) we need to check that the morphisms $\mathrm{ev}_t \circ \alpha, \mathrm{ev}_t \circ \bar{\alpha} \colon A \to D$ define a quasi-homomorphism from A to B in D. First notice that $(\mathrm{ev}_t \circ \alpha)(a) - (\mathrm{ev}_t \circ \bar{\alpha})(a) = (\mathrm{ev}_t \circ (\alpha - \bar{\alpha}))(a)$ is in B because $(\alpha - \bar{\alpha})(a) \in B[0,1]$. Now consider an element $b \in B$. We want to prove that the product $(\mathrm{ev}_t \circ \alpha)(a)b$ is in B. Consider a function $\phi \in B[0,1]$ such that $\mathrm{ev}_t \circ f = b$. Then $(\mathrm{ev}_t \circ \alpha)(a)b = \mathrm{ev}_t \circ (\alpha(a)f)$ and $\alpha(a)f \in B[0,1]$. Similarly, we can prove that $B(\mathrm{ev}_t \circ \alpha)(a) \subseteq B$.

We can now apply (3) and we obtain $E(ev_t \circ \alpha, \mathrm{ev}_t \circ \bar{\alpha}) = E(\mathrm{ev}_t) \circ E(\alpha, \bar{\alpha})$. Since E is diffotopy invariant, $E(\mathrm{ev}_0) = E(\mathrm{ev}_1)$ and thus we conclude the result. □

2.7 $\mathbb{Z}$-graded algebras

In this section, we summarize the results that have been obtained for computing the invariants of $\mathbb{Z}$-graded algebras. We will recall results from the theory of C*-algebras and see how these have been recovered in the case of locally convex algebras.

The origin of this kind of results dates back to the Pimsner-Voiculescu exact sequence which is a classical result for computing the K-theory of C*-algebra crossed product by an automorphism.

Definition 2.29. Let A be a C*-algebra and $\alpha \in \operatorname{Aut}(A)$. The crosed product $A \rtimes_\alpha \mathbb{Z}$ is the universal C*-algebra generated by A and a unitary element u satisfying the relations

$$ua = \alpha(a)u$$

for all $a \in A$.

The algebra $A \rtimes_\alpha \mathbb{Z}$ contains the algebra

$$\{\sum_{i\in\mathbb{Z}} a(i)u^i | a \in C_c(\mathbb{Z}, A)\}$$

as a dense subalgebra. There is a natural grading assigning degree 1 to u and degree 0 to elements of A.

The exact sequence in the following theorem is called the Pimsner-Voiculescu exact sequence.

Theorem 2.30 ([21]). *There is an exact sequence*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{1-\alpha_*} & K_1(A) \end{array}$$

We note that this sequence relates the K-theory of the crossed product $A \rtimes_\alpha \mathbb{Z}$ with that of A. In terms of the grading, it relates the K-theory of the $\mathbb{Z}$-graded algebra to that of the degree 0 subalgebra. This sequence has been used, for instance, to compute the K-theory of the irrational rotation algebras A_θ.

The proof of this theorem that Cuntz gave in [6] uses Kasparov's KK and the Toeplitz extension associated to the crossed product

$$0 \to \mathbb{K} \otimes_\pi A \to \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z}$$

where $\mathcal{T}_\alpha$ is the C*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{T}_{\mathrm{C}^*}$ generated by $A \otimes 1$ and $u \otimes v$ where v is the isometry that generates $\mathcal{T}_{\mathrm{C}^*}$. The kernel $\mathbb{K} \otimes_\pi A$ is equivalent to A in KK. Once the equivalence between $\mathcal{T}_\alpha$ and A is stablished, the theorem will follow.

This kind of sequences have been later constructed for covariance C*-algebras associated to partial automorphisms (see [13]), Cuntz-Pimsner algebras (see [20]) and generalized crossed products (see [1]).

All of these are examples of $\mathbb{Z}$-graded C*-algebras that satisfy certain conditions on the grading. A $\mathbb{Z}$-grading on a C*-algebra B is equivalent to a circle action $\alpha\colon S^1 \to \mathrm{End}(B)$. Given a grading $B = \sum_{n\in\mathbb{Z}} B_n$, we can define the action $\alpha_z(b_n) = z^n b_n$. And given an action α we define the spectral subspaces $B_n = \{b \in B | \alpha_z(b) = z^n b\}$.

Definition 2.31. A circle action $\alpha\colon S^1 \to \mathrm{End}(B)$ is called semisaturated if the spectral spaces B_0 and B_1 generate B as a C*-algebra.

$\mathbb{Z}$-graded algebras that satisfy the condition of being semisaturated are exactly the generalized crossed products (see Theorem 3.1 of [1]). Covariance algebras by a partial automorphism are $\mathbb{Z}$-graded that are semisaturated and satisfy the condition of being *regular* (see [13]).

Theorem 2.32. *Let B be a semisaturated $\mathbb{Z}$-graded algebra. Then there is an exact sequence*

$$\begin{array}{ccccc} K_0(B_1B_{-1}) & \longrightarrow & K_0(B_0) & \xrightarrow{i_*} & K_0(B \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(B_0) & \longleftarrow & K_1(B_1B_{-1}). \end{array}$$

Proof. The statement in the case of regular semisaturated algebras is in theorem 7.1 of [13]. In remark 3.4 of [1] the proof of the theorem is sketched. □

This theorem generalizes the Pimsner-Voiculescu sequence, since a crossed product $A \rtimes_\alpha \mathbb{Z}$ is a semisaturated $\mathbb{Z}$-graded algebra B with $B_1B_{-1} = B_0 = A$.

In the context of locally convex algebras, similar sequences are constructed for smooth crossed products.

Definition 2.33 (See Section 14 of [10])**.** We define the smooth crossed product $A\hat{\rtimes}_\alpha\mathbb{Z}$ where A is a locally convex algebra and $\alpha \in \mathrm{Aut}(A)$ as the complete locally convex algebra generated by A together with an invertible element u satisfying

$$uxu^{-1} = \alpha(x)$$

for all $x \in A$.

We have the following theorem for smooth crossed products.

Theorem 2.34 (Theorem 14.3 in [10]). *For any locally convex algebra D, there is an exact sequence*

$$\begin{array}{ccccc} kk_0^{\text{alg}}(D,A) & \xrightarrow{\cdot(1-kk(\alpha))} & kk_0^{\text{alg}}(D,A) & \xrightarrow{\cdot kk(i)} & kk_0^{\text{alg}}(D,A\hat{\rtimes}_\alpha\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ kk_1^{\text{alg}}(D,A\hat{\rtimes}_\alpha\mathbb{Z}) & \xleftarrow{\cdot kk(i)} & kk_1^{\text{alg}}(D,A) & \xleftarrow{\cdot(1-kk(\alpha))} & kk_1^{\text{alg}}(D,A), \end{array}$$

where i is the inclusion of A into $A\hat{\rtimes}_\alpha\mathbb{Z}$.

In [15], Gabriel and Grensing defined smooth generalized crossed products. These are certain $\mathbb{Z}$-graded locally convex algebras analog to C*-algebra generalized crossed products.

Definition 2.35. A gauge action γ on a locally convex algebra B is a pointwise continuous action of S^1 on B. An element $b \in B$ is called gauge smooth if the map $t \mapsto \gamma_t(b)$ is smooth.

If we have a gauge action on B, then $B_n = \{b \in B | \gamma_z(b) = z^n b,\ \forall z \in S^1\}$ defines a natural $\mathbb{Z}$-grading of B.

Definition 2.36. A smooth generalized crossed product is a locally convex algebra B with an involution and a gauge action such that

- B_0 and B_1 generate B as a locally convex involutive algebra.
- all b are gauge smooth and the map $B \to C^\infty(S^1, B)$ is continuous.

In the same article, 6-term exact sequences for smooth generalized crossed products B that satisfy the condition of being tame smooth are constructed (see definition 18 in [15]). These sequences relate the kk^{alg} invariants of B with the kk^{alg} invariants of their degree 0 subalgebra B_0.

Theorem 2.37 (Theorem 36 in [15]). *Let B be a tame smooth generalized crossed product. For any locally convex algebra D we have a 6-term exact sequence*

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(D,B_0) & \longrightarrow & kk_0^{\mathrm{alg}}(D,B_0) & \longrightarrow & kk_0^{\mathrm{alg}}(D,B) \\ \uparrow & & & & \downarrow \\ kk_1^{\mathrm{alg}}(D,B) & \longleftarrow & kk_1^{\mathrm{alg}}(D,B_0) & \longleftarrow & kk_1^{\mathrm{alg}}(D,B_0), \end{array}$$

and a similar sequence on the other variable.

Remark 2.38. Equivalently, the result of Theorem 2.37 can be seen as the existence of the following exact triangle in $\mathfrak{KK}^{\mathrm{alg}}$

$$SB \to B_0 \to B_0 \to B.$$

Chapter 3

Generalized Weyl algebras

Generalized Weyl algebras were introduced by Bavula (see [2]) and have been amply studied. Examples of generalized Weyl algebras include the Weyl algebra, the quantum Weyl algebra, the quantum plane, the enveloping algebra of $\mathfrak{sl}_2$, $U(\mathfrak{sl}_2)$, and its primitive factors $B_\lambda = U(\mathfrak{sl}_2)/\langle C - \lambda \rangle$ where C is the Casimir element (see Example 4.7 in [17]).

In our context, generalized Weyl algebras provide a family of examples of $\mathbb{Z}$-graded algebras that might not be smooth generalized crossed products or might not satisfy the condition of being tame smooth (and therefore in general they are outside the framework of [15]).

3.1 Definition and properties

In this section, we define generalized Weyl algebras and establish their main properties.

Definition 3.1. Let D be a ring, $\sigma \in \mathrm{Aut}(D)$ and a a central element of D. The generalized Weyl algebra $D(\sigma, a)$ is the algebra generated by x and y over D satisfying

$$xd = \sigma(d)x, \; yd = \sigma^{-1}(d)y, \; yx = a \text{ and } xy = \sigma(a) \tag{3.1}$$

for all $d \in D$.

Examples 3.2. The following are examples of generalized Weyl algebras

1. The Weyl algebra

$$A_1(\mathbb{C}) = \mathbb{C}\langle x, y | xy - yx = 1\rangle$$

is isomorphic to $\mathbb{C}[h](\sigma, h)$, with $\sigma(h) = h - 1$.

2. The quantum Weyl algebra

$$A_q(\mathbb{C}) = \mathbb{C}\langle x, y | xy - qyx = 1\rangle$$

is isomorphic to $\mathbb{C}[h](\sigma, h - 1)$, with $\sigma(h) = qh$.

3. The quantum plane

$$\mathbb{C}\langle x, y | xy = qyx\rangle$$

is isomorphic to $\mathbb{C}[h](\sigma, h)$, with $\sigma(h) = qh$.

4. The primitive quotients of $U(\mathfrak{sl}_2)$ (see Example 3.2 in [2]),

$$B_\lambda = U(\mathfrak{sl}_2)/\langle c - \lambda\rangle, \quad \lambda \in \mathbb{C},$$

are isomorphic to $\mathbb{C}[h](\sigma, P)$, with $\sigma(h) = h - 1$ and $P(h) = -h(h + 1) - \lambda/4$.

5. The quantum weighted projective space or the quantum spindle algebra $\mathcal{O}(\mathbb{WP}_{k,l})$ (see Example 3.8 in [3]) is isomorphic to $\mathbb{C}[h](\sigma, P)$ with $P(h) = h^k \prod_{i=0}^{l-1}(1 - q^{-2i}h)$ and $\sigma(h) = q^{2l}h$.

6. The previous examples are generalized Weyl algebras over $\mathbb{C}[h]$. The enveloping algebra of $\mathfrak{sl}_2$,

$$U(\mathfrak{sl}_2) = \mathbb{C}\langle E, F, H | [E, H] = 2E, [F, H] = -2F, [E, F] = 2H\rangle$$

is isomorphic to $\mathbb{C}[h, c](\sigma, a)$ where $\sigma(h) = h - 1$, $\sigma(c) = c$ and $a = c - h(h+1)$. This case will not be treated in this article since we focus on generalized Weyl algebras over $\mathbb{C}[h]$.

Lemma 3.3. *A generalized Weyl algebra has a $\mathbb{Z}$-grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where $A_0 = D$ and*

$$A_n = \begin{cases} Dy^n & n > 0 \\ Dx^n & n < 0. \end{cases} \tag{3.2}$$

Proof. Consider the grading in $A = D(\sigma, a)$ defined by setting the degree of y equal to 1, the degree of x equal to -1, and the degree of all elements of D equal to 0. That is, the degree of the monomial $\prod_{i=1}^{n} d_i x^{\alpha_i} y^{\beta_i}$, with $d_i \in D$, is equal to $\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \alpha_i$. Since the relations defining A are compatible with the grading, the algebra A is $\mathbb{Z}$-graded.

Now consider the following relations in A. We have

$$\begin{aligned} x^n y^n &= \sigma^n(a)\sigma^{n-1}(a)\dots\sigma(a) \\ y^n x^n &= \sigma^{-(n-1)}(a)\sigma^{-(n-2)}(a)\dots a \end{aligned}$$

Using induction on the length of the monomial $\prod_{i=1}^{n} d_i x^{\alpha_i} y^{\beta_i}$ we prove (3.2). Note that $Dy^n = y^n D$ and $Dx^n = x^n D$. □

In the case of generalized Weyl algebras over $\mathbb{C}[h]$, we have the following result.

Corollary 3.4. *The generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P)$, with $P \in \mathbb{C}[h]$, has a countable basis over $\mathbb{C}$.*

Proof. A basis is given by the elements h^n, $h^n y^m$ and $h^n x^m$ for $n \in \mathbb{N}$, $m \geq 1$. □

There are several ways of writing the same generalized Weyl algebra. The conjugation of σ by an automorphism τ of D gives rise to an isomorphism of generalized Weyl algebras.

Lemma 3.5. *Let σ, τ be automorphisms of D and let a be a central element of D. Then $\tau(a)$ is central in D and*

$$D(\sigma, a) \cong D(\tau\sigma\tau^{-1}, \tau(a)).$$

Proof. Let x' and y' be the generators of $D(\tau\sigma\tau^{-1}, \tau(a))$ over D. There is a morphism $\phi\colon D(\sigma, a) \to D(\tau\sigma\tau^{-1}, \tau(a))$ defined by $x \mapsto x'$, $y \mapsto y'$, $d \mapsto \tau(d)$, for all $d \in D$. We need to check that ϕ is compatible with the relations of (3.1). Using the relations defining $D(\tau\sigma\tau^{-1}, \tau(a))$ we have

$$\begin{aligned} x'\tau(d) &= (\tau\sigma\tau^{-1})(\tau(d))x' = \tau(\sigma(d))x' \\ y'\tau(d) &= (\tau\sigma^{-1}\tau)(\tau(d))y' = \tau(\sigma^{-1}(d))y' \\ x'y' &= \tau(a) \\ y'x' &= (\tau\sigma\tau^{-1})(\tau(a)) = \tau(\sigma(a)). \end{aligned}$$

ϕ^{-1} is defined by $x' \mapsto x$, $y' \mapsto y$, $d \mapsto \tau^{-1}(d)$ for all $d \in D$. □

In the case $D = \mathbb{C}[h]$, we use Lemma 3.5 to write a given generalized Weyl algebra in a canonical form. Any automorphism of $\mathbb{C}[h]$, is of the form $\sigma(h) = qh + h_0$ with $q, h_0 \in \mathbb{C}$ and $q \neq 0$. We have three cases

1. σ is conjugate to id if and only if $\sigma = \mathrm{id}$,
2. if $q = 1$ and $h_0 \neq 0$, then σ is conjugate to $h \mapsto h - 1$,
3. if $q \neq 1$, then σ is conjugate to $h \mapsto qh$.

Combining this with Lemma 3.5, we obtain the following result.

Proposition 3.6 (Compare with Proposition 2.1.1 in [22].)**.** *Let $A = \mathbb{C}[h](\sigma, P)$, with $P \in \mathbb{C}[h]$ and $\sigma(h) = qh + h_0$ with $q, h_0 \in \mathbb{C}$ and $q \neq 0$. The following facts hold.*

1. *If $\sigma = \mathrm{id}$, then $A \cong \mathbb{C}[h, x, y]/(xy - P)$.*
2. *If $q = 1$ and $h_0 \neq 0$ then $A \cong \mathbb{C}[h](\sigma_1, P_1)$ with $\sigma_1(h) = h-1$ and $P_1(h) = P(-h_0 h)$.*
3. *If $q \neq 1$ then $A \cong \mathbb{C}[h](\sigma_1, P_1)$ with $\sigma_1(h) = qh$ and $P_1(h) = P(h - \frac{h_0}{1-q})$.*

By Proposition 3.6, we can assume that $\sigma = \mathrm{id}$, $\sigma(h) = h - 1$ or $\sigma(h) = qh$ for some $q \neq 0$.

Proposition 3.7. *Let $A = \mathbb{C}[h](\sigma, P)$, with $P \in \mathbb{C}[h]$. We have*

1. *if $\sigma(h) = h - 1$ and P is a non-constant polynomial, then $A \cong \mathbb{C}[h](\sigma, P_1)$ with $P_1(0) = 0$,*
2. *if $\sigma(h) = qh$ and P has a nonzero root, then $A \cong \mathbb{C}[h](\sigma, P_1)$ with $P_1(1) = 0$.*

3.2 A faithful representation

Now we construct faithful representations for the generalized Weyl algebras covered in cases (1) and (2) of Proposition 3.7. We define $V_{\mathbb{N}}$ as the vector space of sequences of complex numbers indexed by $\mathbb{N}$. Let $\mathcal{U}_1, \mathcal{U}_{-1}, G \in \mathrm{End}(V_{\mathbb{N}})$ as the shifts to the right and to the left respectively. Note that $\mathcal{U}_{-1}\mathcal{U}_1 = 1, \mathcal{U}_1\mathcal{U}_{-1} = 1 - e_{00}$.

$$\mathcal{U}_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & & \ddots \end{bmatrix} \qquad \mathcal{U}_{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Additionally, we use the following elements $N = \sum_{i\in\mathbb{N}}(-i)e_{i,i}$ and $G = \sum_{i\in\mathbb{N}} q^i e_{i,i}$ for $q \neq 0$ not a root of unity.

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & -2 & 0 & \\ 0 & 0 & 0 & -3 & \\ \vdots & & & & \ddots \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & q & 0 & 0 & \\ 0 & 0 & q^2 & 0 & \\ 0 & 0 & 0 & q^3 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Lemma 3.8. *The following relations are satisfied in* $\mathrm{End}(V_{\mathbb{N}})$.

1. $\mathcal{U}_1 N = (N+1)\mathcal{U}_1$.

2. $\mathcal{U}_{-1} N = (N-1)\mathcal{U}_{-1}$,

3. $\mathcal{U}_1 G = (q^{-1}G)\mathcal{U}_1$,

4. $\mathcal{U}_{-1} G = (qG)\mathcal{U}_{-1}$.

5. *If P is a polynomial and $k \in \mathbb{N}$, then*

$$[P(N)\mathcal{U}_1]^k = \mathcal{U}_1^k P(N-1)P(N-2)\dots P(N-k).$$

6. *If P is a polynomial and $k \in \mathbb{N}$, then*

$$[P(G)\mathcal{U}_1]^k = \mathcal{U}_1^k P(qG)P(q^2G)\dots P(q^kG).$$

Proof. The relations in (1)-(4) are readily checked. For (5), we note that by (1) we have $P(N)\mathcal{U}_1 = \mathcal{U}_1 P(N-1)$ and the result follows by interchanging factors in

$$P(G)\mathcal{U}_1 P(G)\mathcal{U}_1 \dots P(G)\mathcal{U}_1.$$

Item (6) is proved similarly, using $P(G)\mathcal{U}_1 = \mathcal{U}_1 P(qG)$ which follows from (2). □

As a consequence of Lemma 3.8, we obtain that the subalgebras $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathrm{End}(V_{\mathbb{N}})$ generated by $\{\mathcal{U}_1, \mathcal{U}_{-1}, N\}$ and $\{\mathcal{U}_1, \mathcal{U}_{-1}, G\}$, respectively, have countable bases over $\mathbb{C}$ and therefore they are locally convex algebras with the fine topology.

Lemma 3.9. *We have representations for generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P(h))$ in the following cases.*

1. *If $\sigma(h) = h - 1$ and P is a nonzero polynomial with $P(0) = 0$, then there is a faithful representation $\rho : A \to \mathcal{E}_1$ such that*

$$\rho(h) = N,\ \ \rho(x) = \mathcal{U}_{-1} \text{ and } \rho(y) = P(N)\mathcal{U}_1 = \mathcal{U}_1 P(N-1).$$

2. *If $\sigma(h) = qh$ with $q \neq 0$ not a root of unity and $P(1) = 0$, then there is a faithful representation $\rho : A \to \mathcal{E}_2$ such that*

$$\rho(h) = G,\ \ \rho(x) = \mathcal{U}_{-1} \text{ and } \rho(y) = P(G)\mathcal{U}_1 = \mathcal{U}_1 P(qG).$$

Proof. For (1), first we notice that we have an injective homomorphism $\mathbb{C}[h] \hookrightarrow \mathrm{End}(V_{\mathbb{N}})$ defined by $h \mapsto N$.

This homomorphism is injective because all the entries in the diagonal of matrix N are different. With $P(0) = 0$ we will see that the relations of $\mathbb{C}[h](\sigma, P(h))$ hold. To prove this, we use the relations of Lemma 3.8. For a polynomial $\alpha(h) \in \mathbb{C}[h]$ we have

$$\begin{aligned}
\rho(x\alpha(h)) &= \mathcal{U}_{-1}\alpha(N) = \alpha(N-1)\mathcal{U}_{-1} = \rho(\alpha(h-1)x) \\
\rho(y\alpha(h)) &= P(N)\mathcal{U}_1\alpha(N) = \alpha(N+1)P(N)\mathcal{U}_1 = \rho(\alpha(h+1)y) \\
\rho(yx) &= \mathcal{U}_1 P(N-1)\mathcal{U}_{-1} = \mathcal{U}_1\mathcal{U}_{-1}P(N) = (1 - e_{00})P(N) = P(N) = \rho(P(h)) \\
\rho(xy) &= \mathcal{U}_{-1}\mathcal{U}_1 P(N-1) = P(N-1) = \rho(P(h-1))
\end{aligned}$$

We use that $P(0) = 0$ in the third row to guarantee $(1 - e_{00})P(N) = P(N)$.

Now we prove that ρ is injective. Let

$$\alpha = \sum_{n\geq 0} p_n(h)y^n + \sum_{m<0} q_m(h)x^m$$

be an element of A. Then we have

$$\rho(\alpha) = \sum_{n\geq 0} p_n(P(N))(P(N)\mathcal{U}_1)^n + \sum_{m<0} q_m(P(N))\mathcal{U}_{-1}^m.$$

Note that $(P(N)\mathcal{U}_1)^n = Q_n(N)\mathcal{U}_1^n$ where $Q_n(N) = P(N)P(N+1)\dots P(N+(n-1))$. Therefore if $\rho(\alpha) = 0$ then $q_m = 0$ and because $Q_n \neq 0$, we have $p_n = 0$. Therefore $\alpha = 0$ and so ρ is injective.

(2) is proved in a similar way: we have an injective homomorphism $\mathbb{C}[h] \hookrightarrow \mathrm{End}(V_{\mathbb{N}})$ defined by $h \mapsto G$. This homomorphism is injective because $q \neq 0$ not a root of unity imply that all the entries in the diagonal of matrix G are different. Using $P(1) = 0$, it is easy to see that the relations of $D(\sigma, a)$ hold. We also need to use the relations of Lemma 3.8. We prove that ρ is injective in a similar way. In this case we note that $(P(G)\mathcal{U}_1)^n = Q_n(G)\mathcal{U}_1^n$ where $Q_n(G) = P(G)P(q^{-1}G)\dots P(q^{-(n-1)}G)$.

□

Remark 3.10. Lemma 3.9 covers every noncommutative generalized Weyl algebra over $\mathbb{C}[h]$ except the following cases

(i) P constant. The case $P = 0$ is treated in Proposition 4.20. In the case P is a nonzero constant polynomial, we follow the construction of [15]. We treat this case in Proposition 4.19.

(ii) $\sigma(h) = qh + h_0$, with q not a root of unity and P having only $\frac{h_0}{1-q}$ as a root. We treat this case in Proposition 4.17.

(iii) $\sigma(h) = qh + h_0$ with $q \neq 1$ a root of unity. This case remains open.

3.3 Relation to smooth generalized crossed products

In [15], Gabriel and Grensing define smooth generalized crossed products. These are involutive locally convex algebras analog to C^*-algebra generalized crossed products in [1]. In the same article [15], sequences analog to the Pimsner-Voiculescu exact sequence are constructed for smooth generalized crossed products that are tame smooth (see definition 18 in [15]).

Definition 3.11. A gauge action γ on a locally convex algebra B is a pointwise continuous action of S^1 on B. An element $b \in B$ is called gauge smooth if the map $t \mapsto \gamma_t(b)$ is smooth.

If we have a gauge action on B, then $B_n = \{b \in B | \gamma_t(b) = t^n b, \ \forall t \in S^1\}$ defines a natural $\mathbb{Z}$-grading of B.

Definition 3.12. A smooth generalized crossed product is a locally convex algebra B with an involution and a gauge action such that

- B_0 and B_1 generate B as a locally convex involutive algebra,
- all b are gauge smooth and the induced map $B \to C^\infty(S^1, B)$ is continuous.

Generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ are locally convex algebras when given the fine topology. When $P \in \mathbb{R}[h]$ and q and h_0 are real, they have an involution defined by $y^* = x$, $x^* = y$ and d^* obtained by conjugating the coefficients of $d \in \mathbb{C}[h]$. There is an action of S^1 defined by $\gamma_t(\omega_n) = t^n \omega_n$ for $\omega_n \in A_n$. In this case, generalized Weyl algebras over $\mathbb{C}[h]$ are smooth generalized crossed products.

Remark 3.13. In the case when $P \in \mathbb{R}[h]$ and q and h_0 are real, generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ are only tame smooth when P is a nonzero constant polynomial (see definition 18 in [15]). If P is non-constant, we have $A_1 A_{-1} = (P) \subsetneq A_0 = \mathbb{C}[h]$. This implies that A is not tame smooth because tame smooth generalized crossed products B have a frame in degree 1 which implies that $B_1 B_{-1} = B_0$.

3.4 The Toeplitz extension of a generalized Weyl algebra

We define the Toeplitz algebra associated to a $\mathbb{Z}$-graded locally convex algebra. This definition is akin to the one given for smooth generalized crossed products in [15].

Definition 3.14. Let A be a $\mathbb{Z}$-graded locally convex algebra. We define $\mathcal{T}_A$ to be the closed subalgebra of $\mathcal{T} \otimes_\pi A$ generated by $S^i \otimes A_i$ and $S^{*j} \otimes A_{-j}$ for all $i \geq 0, j \geq 1$.

Note that in the case that A is generated by A_0, A_1 and A_{-1} then $\mathcal{T}_A$ is generated by $1 \otimes A_0, S \otimes A_1$ and $S^* \otimes A_{-1}$.

We tensor the linearly split extension $0 \to \mathcal{K} \to \mathcal{T} \to C^\infty(S^1) \to 0$ with A to obtain

$$0 \to \mathcal{K} \otimes_\pi A \to \mathcal{T} \otimes_\pi A \xrightarrow{p} C^\infty(S^1) \otimes_\pi A \to 0. \tag{3.3}$$

which is still a linearly split extension.

Proposition 3.15. *Let A be a generalized Weyl algebra $\mathbb{C}[h](\sigma, P)$. Then there is a linearly split extension*

$$0 \to \Lambda_A \xrightarrow{\iota} \mathcal{T}_A \xrightarrow{\bar{p}} A \to 0$$

where Λ_A is the closure of the ideal $\bigoplus_{i,j\geq 0} e_{i,j} \otimes A_{i+1}A_{-(j+1)}$ in $\mathcal{K} \otimes_\pi A$, ι is the inclusion of Λ_A in $\mathcal{T}_A$ and $\bar{p}$ is the restriction of p to $\mathcal{T}_A$.

Proof. Because of the splitting of the sequence, the image of $\mathcal{T}_A$ in $C^\infty(S^1) \otimes_\pi A$ is the closed algebra generated by $1 \otimes A_0$, $z \otimes A_1$ and $z^{-1} \otimes A_{-1}$ in $C^\infty(S^1) \otimes_\pi A$. We map Im $\bar{p} \to A$ via the restriction of $\mathrm{ev}_1 \otimes 1_A : C^\infty(S^1) \otimes_\pi A \to \mathbb{C} \otimes_\pi A \cong A$. The inverse is given by $A \to \mathrm{Im}\ \bar{p}$, $\sum a_n \mapsto z^n \otimes a_n$, which is continuous because A is endowed with the fine topology.

The proof that the kernel of $\bar{p}$ is the closure of the ideal $\bigoplus_{i,j\geq 0} e_{i,j} \otimes A_{i+1}A_{-(j+1)}$ in $\mathcal{K} \otimes_\pi A$ is the same as that of Proposition 23 in [15].

□

Although by construction the elements of Λ_A could be infinite sums we prove that in fact these sums are finite.

Lemma 3.16. *The elements of Λ_A are finite sums*

$$\sum_{i,j\geq 0} e_{i,j}\otimes y^{i+1}P_{i,j}(h)x^{j+1}$$

with $P_{i,j}(h)\in\mathbb{C}[h]$.

Proof. Let $\omega=\sum_{i,j\geq 0}^{\infty} e_{i,j}\otimes a_{i+1,-(j+1)}$ be an element of Λ_A. We claim that only finitely many of the $a_{i+1,-(j+1)}$ are non-zero. Since $\mathcal{K}\otimes_\pi A=\mathcal{K}\otimes A$, an element $\omega\in\mathcal{K}\otimes_\pi A$ can be written as $\omega=\sum_{t=1}^{M} m^{(t)}\otimes f^{(t)}$ where $m^{(t)}\in\mathcal{K}$ and $f^{(t)}\in A$. Because we are dealing with finitely many $f^{(t)}$, we can assume that there is an $N>0$ such that the degree of all homogeneous components of $f^{(t)}$ in A is bounded between $-N$ and N. By Lemma 3.3, we can write

$$f^{(t)}=\sum_{k=0}^{N} P_k^{(t)}(h)y^k+\sum_{k=1}^{N} P_{-k}^{(t)}(h)x^k.$$

Let D be the maximum degree of all polynomials $P_k^{(t)}$. If $m^{(t)}=\sum_{i,j\geq 0}^{\infty} c_{i,j}^{(t)}e_{i,j}$ we have

$$\begin{aligned}
\omega &= \sum_{t=1}^{M}\left(\sum_{i,j\geq 0}^{\infty} c_{i,j}^{(t)}e_{i,j}\right)\otimes f^{(t)}\\
&= \sum_{i,j\geq 0}^{\infty} e_{i,j}\otimes\left(\sum_{t=1}^{M} c_{i,j}^{(t)}f^{(t)}\right)\\
&= \sum_{i,j\geq 0}^{\infty} e_{i,j}\otimes\left[\sum_{k=0}^{N}\left(\sum_{t=1}^{M} c_{i,j}^{(t)}P_k^{(t)}(h)\right)y^k+\sum_{k=1}^{N}\left(\sum_{t=1}^{M} c_{i,j}^{(t)}P_{-k}^{(t)}(h)\right)x^k\right].
\end{aligned}$$

We notice that the degree of $a_{i+1,-(j+1)}\in A_{i+1}A_{-(j+1)}$ in A is $i-j$, therefore $a_{i+1,-(j+1)}\neq 0$ only if $|i-j|\leq N$. For fixed i and j we have $a_{i+1,-(j+1)}=q(h)y^{i+1}x^{j+1}$ with $q(h)\in\mathbb{C}[h]$. Let n be the degree of the polynomial P that defines A. From the relations defining $A=\mathbb{C}[h](\sigma,P)$, we can deduce $y^kx^k=\prod_{s=0}^{k-1}P(\sigma^{-s}(h))$ which is a polynomial of degree nk. We then have

$$a_{i+1,-(j+1)}=\begin{cases} Q(h)y^{i-j} & ,i\geq j\\ Q(h)x^{j-i} & ,i<j,\end{cases}$$

where $\deg Q\geq n\min\{i+1,j+1\}$. Since we must have $\deg Q\leq D$ in order to have $a_{i+1,-(j+1)}\neq 0$, then if $a_{i+1,-(j+1)}\neq 0$ we have $i+1\leq\frac{D}{n}$ or $j+1\leq\frac{D}{n}$ and using $|i-j|\leq N$, we can conclude $a_{i+1,-(j+1)} \neq 0$ only if $i + j = i - j + 2j = i - j + 2i \leq N + 2(\frac{D}{n} - 1)$. This implies that $a_{i+1,-(j+1)} \neq 0$ for finitely many i, j. $\square$

Chapter 4

$kk^{\mathbf{alg}}$ invariants of generalized Weyl algebras

In this chapter, we compute the isomorphism class in $\mathfrak{KK}^{\mathrm{alg}}$ of generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ where $\sigma(h) = qh + h_0$ is an automorphism of $\mathbb{C}[h]$ and $P \in \mathbb{C}[h]$. We summarize our results in the table below.

Conditions		Results		Observation
P is constant	$P = 0$	$A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}$	Prop 4.20	A $\mathbb{N}$-graded
	$P \neq 0$	$A \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C} \oplus \mathbb{C}$	Prop 4.19	
P is nonconstant with r distinct roots	q not a root of unity	$A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}^r$	Thm 4.13 Prop 4.17	
	$q = 1$ and $h_0 \neq 0$	$A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}^r$	Thm 4.13	

In the case where P is a non-constant polynomial, A might not be a smooth generalized crossed product, and if it is, it is not tame smooth so we cannot apply the results of [15] directly. In this case we follow the methods of [10] and [15] to obtain

$$\Lambda_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_1 A_{-1} \quad \text{(Theorem 4.1)} \qquad \text{and} \quad \mathcal{T}_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0 \quad \text{(Theorem 4.9)}$$

in the cases where P is non-constant and

- $q = 1$ and $h_0 \neq 0$ or

- q is not a root of unity and P has a root different from $\frac{h_0}{q-1}$.

With these isomorphisms we construct in Theorem 4.10 an exact triangle

$$SA \to A_1 A_{-1} \xrightarrow{0} A_0 \to A$$

in the triangulated category $(\mathfrak{KK}^{\mathrm{alg}}, S)$ (see Proposition 2.20). This implies

$$A = A_0 \oplus S(A_1 A_{-1}).$$

In Proposition 4.12, we prove that $A_1 A_{-1} \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C}^{r-1}$, and since by Lemma 4.16 we know that $A_0 \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}$, we obtain our main result Theorem 4.13: in these cases $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}^r$.

We also determine the $\mathfrak{KK}^{\mathrm{alg}}$-class of A when A is $\mathbb{N}$-graded. In this case Lemma 4.16 gives us $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0$. This is the case when

- P is nonconstant, q is not a root of unity and P has only $\frac{h_0}{q-1}$ as a root or

- $P = 0$.

In both cases we obtain $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}$ in Propositions 4.20 and 4.17.

If P is a nonzero real constant, A is a tame smooth generalized crossed product and the results from [15] apply. In case P is a non zero constant, the proofs of [15] still hold and we obtain $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C} \oplus \mathbb{C}$ in Proposition 4.19.

4.1 The case where P is a non-constant polynomial

We consider the short exact sequence

$$0 \to \Lambda_A \to \mathcal{T}_A \to A \to 0$$

from Proposition 3.15, where $\mathcal{T}_A$ is the Toeplitz algebra of A. This sequence yields an exact triangle

$$SA \to \Lambda_A \to \mathcal{T}_A \to A$$

in the triangulated category $(\mathfrak{KK}^{\mathrm{alg}}, S)$.

In order to apply Lemma 3.9, we need to consider generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$ where P is non-constant and

- $q = 1$ and $h_0 \neq 0$ or
- q is not a root of unity and P has a root different from $\frac{h_0}{1-q}$.

By Propositions 3.6 and 3.7, in order to cover these cases, it suffices to consider the following two cases:

- $\sigma(h) = h - 1$ and P is a non-constant polynomial with $P(0) = 0$.
- $\sigma(h) = qh$ where q is not a root of unity and P is a non-constant polynomial with $P(1) = 0$.

We first consider generalized Weyl algebras satisfying these assumptions and treat the case where q is not a root of unity and P has only $\frac{h_0}{1-q}$ as a root separately in Proposition 4.17.

Define $j_1 : A_1A_{-1} \to \Lambda_A$ by $j_1(a) = e_{00} \otimes a$. We embed Λ_A in a suitable algebra so that we can construct a Morita equivalence to its subalgebra $j_1(A_1A_{-1}) = e_{00} \otimes A_1A_{-1}$.

Now, we show $\Lambda_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_1A_{-1}$.

Theorem 4.1. *There is a Morita equivalence between Λ_A and $j_1(A_1A_{-1})$, therefore there is an invertible element $\theta \in kk^{\mathrm{alg}}(\Lambda_A, A_1A_{-1})$ which is an inverse of $kk(j_1)$.*

Proof. First, write the proof in the case $\sigma(h) = h - 1$ and $P(0) = 0$. Consider the representation

$$\rho : A = \mathbb{C}[h](\sigma, P) \to \mathcal{E}_1$$

from item (1) in Lemma 3.9, that is $\rho(h) = N$, $\rho(x) = \mathcal{U}_{-1}$ and $\rho(y) = P(N)\mathcal{U}_1$. Tensoring with $1_{\mathcal{T}}$ we obtain an injective morphism $1_{\mathcal{T}} \otimes \rho : \mathcal{T} \otimes A \to \mathcal{T} \otimes \mathcal{E}_1$ which restricts to an injective morphism

$$\bar{\rho} : \Lambda_A \hookrightarrow \mathcal{T} \otimes \mathcal{E}_1.$$

The Morita equivalence is given by $\xi_i = \xi'_i = e_{i,0} \otimes \mathcal{U}_1^i$ and $\eta_j = \eta'_j = e_{0,j} \otimes \mathcal{U}_{-1}^i$. We check that these sequences satisfy the conditions in Definition 2.21 and Proposition 2.22.

First, we establish that ξ_i, η_j defines a Morita context between Λ_A and $e_{00} \otimes A_1A_{-1}$ according with Definition 2.21. Let $w = \sum e_{i,j} \otimes y^{i+1}P_{i,j}(h)x^{j+1}$ be an element of Λ_A.

1. $\eta_j\bar{\rho}(w)\xi_i \in e_{00} \otimes A_1A_{-1}$. We have

$$\eta_j\bar{\rho}(w)\xi_i = e_{00} \otimes \mathcal{U}_{-1}^j[P(N)\mathcal{U}_1]^{j+1}P_{j,i}(N)\mathcal{U}_{-1}^{i+1}\mathcal{U}_1^i$$

   Using item (5) from Lemma 3.8, we can write $(P(N)\mathcal{U}_1)^{j+1} = \mathcal{U}_1^{j+1}R_{j+1}(N)$ where

$$R_{j+1}(N) = P(\sigma(N))\dots P(\sigma^{j+1}(N)).$$

   Naming $R'_{j+1}(N) = P(\sigma^2(N))\dots P(\sigma^{j+1}(N))$ we have

$$R_{j+1}(N) = P(\sigma(N))R'_{j+1}(N).$$

   Therefore

$$\begin{aligned}
\eta_j\bar{\rho}(w)\xi_i &= e_{00} \otimes \mathcal{U}_1R_{j+1}(N)P_{j,i}(N)\mathcal{U}_{-1} \\
&= e_{00} \otimes \mathcal{U}_1P(\sigma(N))R'_{j+1}(N)P_{j,i}(N)\mathcal{U}_{-1} \\
&= e_{00} \otimes P(N)\mathcal{U}_1R'_{j+1}(N)P_{j,i}(N)\mathcal{U}_{-1} \\
&= \bar{\rho}(e_{00} \otimes yR'_{j+1}(h)P_{j,i}(h)x) \in \bar{\rho}(e_{00} \otimes A_1A_{-1}).
\end{aligned}$$

2. The terms $\eta_j\bar{\rho}(w)\xi_i$ are rapidly decreasing. This is because the elements of Λ_A are finite sums.

3. $(\sum \xi_i\eta_i)\bar{\rho}(w) = \bar{\rho}(w)$. We have

$$\begin{aligned}
\left(\sum \xi_i\eta_i\right)\bar{\rho}(w) &= \left(\sum e_{i,i} \otimes \mathcal{U}_1^i\mathcal{U}_{-1}^i\right)\left(\sum e_{k,l} \otimes (\mathcal{U}_1P(N))^{k+1}P_{k,l}(N)U_{-1}^{l+1}\right) \\
&= \sum e_{k,l} \otimes \mathcal{U}_1^k\mathcal{U}_{-1}^k\mathcal{U}_1^{k+1}R_{k+1}(N)P_{k,l}(N)\mathcal{U}_{-1}^{l+1} \\
&= \sum e_{k,l} \otimes \mathcal{U}_1^{k+1}R_{k+1}(N)P_{k,l}(N)\mathcal{U}_{-1}^{l+1} \\
&= \bar{\rho}(w)
\end{aligned}$$

Now we check the conditions of Proposition 2.22. We show that $\bar{\rho}(w)\xi_k\xi'_l$ and $\eta'_k\eta'_l\bar{\rho}(w)$ are still elements of $\bar{\rho}(\Lambda_A)$.

$$\bar{\rho}(w)\xi_k\xi'_l = \left(\sum e_{i,j} \otimes (\mathcal{U}_1P(N))^{i+1}P_{i,j}(N)U_{-1}^{j+1}\right)(e_{k,0} \otimes \mathcal{U}_1^k)(e_{l,0} \otimes \mathcal{U}_1^l)$$

which is 0 unless $l = 0$ and in that case we obtain

$$\begin{aligned}\bar{\rho}(w)\xi_k\xi'_l &= \sum e_{i,0} \otimes (\mathcal{U}_1 P(N))^{i+1} P_{i,k}(N) U_{-1} \\ &= \bar{\rho}\left(\sum e_{i,0} \otimes y^{i+1} P_{i,k}(h) x\right) \in \bar{\rho}(\Lambda_A)\end{aligned}$$

and similarly we compute

$$\eta'_k \eta_l \bar{\rho}(w) = (e_{0,k} \otimes \mathcal{U}_{-1}^k)(e_{0,l} \otimes \mathcal{U}_{-1}^l)\left(\sum e_{i,j} \otimes (\mathcal{U}_1 P(N))^{i+1} P_{i,j}(N) U_{-1}^{j+1}\right)$$

which is 0 unless $k = 0$ and in that case we obtain

$$\begin{aligned}\bar{\rho}(w)\xi_k\xi'_l &= \sum e_{0,j} \otimes \mathcal{U}_{-1}^l (\mathcal{U}_1 P(N))^{l+1} P_{l,j}(N) U_{-1}^{j+1} \\ &= \bar{\rho}\left(\sum e_{0,j} \otimes y R'_{l+1}(h) P_{l,j}(h) x^{j+1}\right) \in \bar{\rho}(\Lambda_A).\end{aligned}$$

The Morita context from $e_{00} \otimes A_1 A_{-1}$ to Λ_A is defined by (ξ'_i, η'_j). So far we have proved $kk((\xi'_i), (\eta'_j)) \circ kk((\xi_i), (\eta_j)) = 1_{\Lambda_A}$. Let $z = e_{00} \otimes y P_{0,0} x \in e_{00} \otimes A_1 A_{-1}$. Then $\bar{\rho}(z)\xi'_l\xi_k = \bar{\eta}_l \eta'_k \rho(z) = 0$ unless $l = k = 0$ and in this case $\bar{\rho}(z)\xi'_0\xi_0 = \bar{\rho}(z)\eta_0\eta'_0 = \bar{\rho}(z)$. Thus we have $kk((\xi_i), (\eta_j)) \circ kk((\xi'_i), (\eta'_j)) = 1_{e_{00} \otimes A_1 A_{-1}}$.

The proof of the case where $\sigma(h) = qh$ and $P(1) = 0$ is quite similar. Consider the representation

$$\rho : A = \mathbb{C}[h](\sigma, P) \to \mathcal{E}_2$$

from item (2) in Lemma 3.9, that is $\rho(h) = G$, $\rho(x) = \mathcal{U}_{-1}$ and $\rho(y) = P(G)\mathcal{U}_1$. Again, tensoring with $1_{\mathcal{T}}$ we obtain an injective morphism $1_{\mathcal{T}} \otimes \rho : \mathcal{T} \otimes A \to \mathcal{T} \otimes \mathcal{E}_2$ which restricts to an injective morphism

$$\bar{\rho} : \Lambda_A \hookrightarrow \mathcal{T} \otimes \mathcal{E}_2.$$

The Morita equivalence is given by $\xi_i = \xi'_i = e_{i,0} \otimes \mathcal{U}_1^i$ and $\eta_j = \eta'_j = e_{0,j} \otimes \mathcal{U}_{-1}^i$. Let $\omega \in \Lambda_A$ be as above. Again, we check that these sequences satisfy the conditions in Definition 2.21 and Proposition 2.22.

1. $\eta_j \bar{\rho}(w)\xi_i \in e_{00} \otimes A_1 A_{-1}$. We have

$$\eta_j \bar{\rho}(w)\xi_i = e_{00} \otimes \mathcal{U}_{-1}^j [P(Q)\mathcal{U}_1]^{j+1} P_{j,i}(Q) \mathcal{U}_{-1}^{i+1} \mathcal{U}_1^i$$

This time, we use item (6) from Lemma 3.8. We can write $(P(Q)\mathcal{U}_1)^{j+1} = \mathcal{U}_1^{j+1}T_{j+1}(Q)$ where

$$T_{j+1}(Q) = P(\sigma(Q))\dots P(\sigma^{j+1}(Q)).$$

Naming $T'_{j+1}(Q) = P(\sigma^2(Q))\dots P(\sigma^{j+1}(Q))$ we have

$$T_{j+1}(Q) = P(\sigma(Q))T'_{j+1}(Q).$$

Therefore

$$\begin{aligned}
\eta_j\bar{\rho}(w)\xi_i &= e_{00}\otimes\mathcal{U}_1T_{j+1}(Q)P_{j,i}(Q)\mathcal{U}_{-1}\\
&= e_{00}\otimes\mathcal{U}_1P(\sigma(Q))T'_{j+1}(Q)P_{j,i}(Q)\mathcal{U}_{-1}\\
&= e_{00}\otimes P(Q)\mathcal{U}_1T'_{j+1}(Q)P_{j,i}(Q)\mathcal{U}_{-1}\\
&= \bar{\rho}(e_{00}\otimes yT'_{j+1}(h)P_{j,i}(h)x)\in\bar{\rho}(e_{00}\otimes A_1A_{-1}).
\end{aligned}$$

2. The terms $\eta_j\bar{\rho}(w)\xi_i$ are rapidly decreasing. This is because the elements of Λ_A are finite sums.

3. $(\sum\xi_i\eta_i)\bar{\rho}(w) = \bar{\rho}(w)$. We have

$$\begin{aligned}
\left(\sum\xi_i\eta_i\right)\bar{\rho}(w) &= \left(\sum e_{i,i}\otimes\mathcal{U}_1^i\mathcal{U}_{-1}^i\right)\left(\sum e_{k,l}\otimes(\mathcal{U}_1P(Q))^{k+1}P_{k,l}(Q)U_{-1}^{l+1}\right)\\
&= \sum e_{k,l}\otimes\mathcal{U}_1^k\mathcal{U}_{-1}^k\mathcal{U}_1^{k+1}T_{k+1}(Q)P_{k,l}(Q)\mathcal{U}_{-1}^{l+1}\\
&= \sum e_{k,l}\otimes\mathcal{U}_1^{k+1}T_{k+1}(Q)P_{k,l}(Q)\mathcal{U}_{-1}^{l+1}\\
&= \bar{\rho}(w)
\end{aligned}$$

Finally, we check the conditions of Proposition 2.22 in this case. We show that $\bar{\rho}(w)\xi_k\xi'_l$ and $\eta'_k\eta'_l\bar{\rho}(w)$ are still elements of $\bar{\rho}(\Lambda_A)$.

$$\bar{\rho}(w)\xi_k\xi'_l = \left(\sum e_{i,j}\otimes(\mathcal{U}_1P(Q))^{i+1}P_{i,j}(Q)U_{-1}^{j+1}\right)(e_{k,0}\otimes\mathcal{U}_1^k)(e_{l,0}\otimes\mathcal{U}_1^l)$$

which is 0 unless $l=0$ and in that case we obtain

$$\begin{aligned}
\bar{\rho}(w)\xi_k\xi'_l &= \sum e_{i,0}\otimes(\mathcal{U}_1P(Q))^{i+1}P_{i,k}(Q)U_{-1}\\
&= \bar{\rho}\left(\sum e_{i,0}\otimes y^{i+1}P_{i,k}(h)x\right)\in\bar{\rho}(\Lambda_A).
\end{aligned}$$

Next, we compute

$$\eta'_k \eta_l \bar{\rho}(w) = (e_{0,k} \otimes \mathcal{U}_{-1}^k)(e_{0,l} \otimes \mathcal{U}_{-1}^l) \left(\sum e_{i,j} \otimes (\mathcal{U}_1 P(Q))^{i+1} P_{i,j}(Q) U_{-1}^{j+1} \right)$$

which is 0 unless $k = 0$ and in that case we obtain

$$\begin{aligned} \bar{\rho}(w)\xi_k \xi'_l &= \sum e_{0,j} \otimes \mathcal{U}_{-1}^l (\mathcal{U}_1 P(Q))^{l+1} P_{l,j}(Q) U_{-1}^{j+1} \\ &= \bar{\rho} \left(\sum e_{0,j} \otimes y T'_{l+1}(h) P_{l,j}(h) x^{j+1} \right) \in \bar{\rho}(\Lambda_A). \end{aligned}$$

We have proved $kk((\xi'_i), (\eta'_j)) \circ kk((\xi_i), (\eta_j)) = 1_{\Lambda_A}$. Let $z = e_{00} \otimes y P_{0,0}(h) x \in e_{00} \otimes A_1 A_{-1}$. Then $\bar{\rho}(z)\xi'_l \xi_k = \bar{\eta}_l \eta'_k \rho(z) = 0$ unless $l = k = 0$ and in this case $\bar{\rho}(z)\xi'_0 \xi_0 = \bar{\rho}(z)\eta_0 \eta'_0 = \bar{\rho}(z)$. Thus we have $kk((\xi_i), (\eta_j)) \circ kk((\xi'_i), (\eta'_j)) = 1_{e_{00} \otimes A_1 A_{-1}}$.

□

Next, we show $\mathcal{T}_A \cong_{\mathfrak{KK}^{\mathrm{alg}}} A_0$. Define $j_0 : A_0 \to \mathcal{T}_A$ by $j_0(a) = 1 \otimes a$. We show that this inclusion induces an invertible element $kk(j_0) \in kk_0^{\mathrm{alg}}(A_0, \mathcal{T}_A)$.

Lemma 4.2. *There is a quasihomomorphism* $(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) : \mathcal{T}_A \rightrightarrows \mathcal{T} \otimes A \triangleright \mathcal{C}$*, where* $\mathcal{C}$ *is the closure of* $\bigoplus_{i,j \geq 0} e_{i,j} \otimes A_i A_{-j}$ *in* $\mathcal{K} \otimes_\pi A$*. Here* $\mathrm{Ad}(S \otimes 1)$ *is the restriction of* $\mathrm{Ad}(S \otimes 1) : \mathcal{T} \otimes A \to \mathcal{T} \otimes A$ *defined by* $x \mapsto (S \otimes 1)x(S^* \otimes 1)$.

Remark 4.3. By an argument similar to the proof of Lemma 3.16 we can conclude that $\bigoplus_{i,j \geq 0} e_{i,j} \otimes A_i A_{-j}$ is closed.

Proof. We have $A_i A_{-j} A_j A_{-k} \subseteq A_i A_{-k}$ because $A_{-j} A_j \subseteq A_0$, therefore $\mathcal{C}$ is a subalgebra. To prove that the pair $(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$ defines a quasihomomorphism we check the conditions on the generators. It is clear that $(1 \otimes A_0)\mathcal{C}$, $(S \otimes A_1)\mathcal{C}$ and $(S^* \otimes A_{-1})\mathcal{C}$ are subsets of $\mathcal{C}$. Now we let $a_i \in A_i$ and we check

$$\begin{aligned} (\mathrm{id} - \mathrm{Ad}(S \otimes 1))(1 \otimes a_0) &= e \otimes a_0 \in \mathcal{C} \\ (\mathrm{id} - \mathrm{Ad}(S \otimes 1))(S \otimes a_1) &= Se \otimes a_1 \in \mathcal{C} \\ (\mathrm{id} - \mathrm{Ad}(S \otimes 1))(S^* \otimes a_{-1}) &= eS^* \otimes a_{-1} \in \mathcal{C}. \end{aligned}$$

□

Define $i_0 : A_0 \to \mathcal{C}$ by $i_0(a) = e_{00} \otimes a$.

Proposition 4.4. *There is a Morita equivalence between $\mathcal{C}$ and $i_0(A_0)$. Therefore there is an invertible element $\kappa \in kk^{\mathrm{alg}}(\mathcal{C}, A_0)$.*

Proof. Using Lemma 3.9, we think of $\mathcal{C}$ represented in $\mathcal{T} \otimes \mathcal{E}$ (where $\mathcal{E} = \mathcal{E}_1$ if $q = 1$ and $\mathcal{E} = \mathcal{E}_2$ if $q \neq 1$). The Morita equivalence is given by $\xi_i = \xi_i' = e_{i,0} \otimes \mathcal{U}_1^i$ and $\eta_j = \eta_j' = e_{0,j} \otimes \mathcal{U}_{-1}^i$. The proof that these elements determine a Morita equivalence is similar to the proof of Theorem 4.1. We define $\kappa = kk((\xi_i), (\eta_j)) \in kk_0^{\mathrm{alg}}(\mathcal{C}, A_0)$. □

Proposition 4.5. *Let $\kappa \in kk(\mathcal{C}, A_0)$ as in Proposition 4.4, then*

$$\kappa \circ kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) \circ kk(j_0) = 1_{A_0}.$$

This implies that $kk(j_0)$ has a left inverse and that $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$ has a right inverse.

Proof. We have

$$(\mathrm{id} - \mathrm{Ad}(S \otimes 1))(j_0(a_0)) = e_{00} \otimes a_0,$$

thus $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) \circ kk(j_0) = kk(i_0)$. By Proposition 4.4, $\kappa \circ kk(i_0) = 1_{A_0}$. □

To show that $kk(j_0)$ is invertible, we construct a left inverse for $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$. In order to do this, we construct a diffotopic family of quasihomomorphisms between $\mathcal{T}_A$ and a subalgebra $\bar{\mathcal{C}}$ of $(\mathcal{T} \otimes_\pi \mathcal{T}) \otimes A$ and prove that $\bar{\mathcal{C}}$ is Morita equivalent to $\mathcal{T}_A$. To construct this diffotopic family we use the following diffotopy.

Recall the diffotopy

$$\phi_t : \mathcal{T} \to \mathcal{T} \otimes_\pi \mathcal{T}$$

from Lemma 1.36. The images of S and S^* are

$$\phi_t(S) = S^2 S^* \otimes 1 + f(t)(e \otimes S) + g(t)(Se \otimes 1)$$

$$\phi_t(S^*) = SS^{*2} \otimes 1 + \overline{f(t)}(e \otimes S^*) + \overline{g(t)}(eS^* \otimes 1)$$

where $f, g \in \mathbb{C}[0,1]$ are such that $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ and $g(1) = 0$. Note that $\phi_0(S) = S \otimes 1$ and $\phi_1(S) = S^2 S^* \otimes 1 + e \otimes S$.

Consider the map $\Phi_t = \phi_t \otimes \mathrm{id}_A : \mathcal{T} \otimes A \to (\mathcal{T} \otimes_\pi \mathcal{T}) \otimes A$ where ϕ_t is the diffotopy of Lemma 1.36. Since $\phi_0(S) = S \otimes 1$, then $\Phi_0(x \otimes a) = x \otimes 1 \otimes a$.

Lemma 4.6. *There is a diffotopic family of quasihomomorphisms*

$$(\Phi_t, \Phi_0 \circ \mathrm{Ad}(S \otimes 1)) : \mathcal{T}_A \rightrightarrows (\mathcal{T} \otimes_\pi \mathcal{T}) \otimes A \triangleright \bar{\mathcal{C}}.$$

Here $\bar{\mathcal{C}}$ is the closure of

$$\bigoplus_{i,j,p,q \in \mathbb{N}} e_{i,j} \otimes S^p S^{*q} \otimes A_{i+p} A_{-(j+q)}$$

in $(\mathcal{K} \otimes_\pi \mathcal{T}) \otimes A$

Proof. We check that $(\Phi_t, \Phi_0 \circ \mathrm{Ad}(S \otimes 1))$ define quasihomomorphisms using the generators of $\mathcal{T}_A$. First we note that $\Phi_t(1 \otimes A_0)\bar{\mathcal{C}}$, $\Phi_t(S \otimes A_1)\bar{\mathcal{C}}$ and $\Phi_t(S^* \otimes A_{-1})\bar{\mathcal{C}}$ are subsets of $\bar{\mathcal{C}}$. Finally, we compute

$$\begin{aligned}
(\Phi_t - \Phi_0 \circ \mathrm{Ad}(S \otimes 1))(1 \otimes a_0) &= e \otimes 1 \otimes a_0 \in \bar{\mathcal{C}} \\
(\Phi_t - \Phi_0 \circ \mathrm{Ad}(S \otimes 1))(S \otimes a_1) &= f(t)(e \otimes S \otimes a_1) + g(t)(Se \otimes 1 \otimes a_1) \in \bar{\mathcal{C}} \\
(\Phi_t - \Phi_0 \circ \mathrm{Ad}(S \otimes 1))(S^* \otimes a_{-1}) &= \bar{f}(t)(e \otimes S^* \otimes a_{-1}) + \bar{g}(t)(eS^* \otimes 1 \otimes a_{-1}) \in \bar{\mathcal{C}}.
\end{aligned}$$

□

Lemma 4.7. *The elements of $\bar{\mathcal{C}}$ are finite sums*

$$\sum_{i,j,p,q \geq 0} e_{i,j} \otimes S^p S^{*q} \otimes y^{i+p} P_{i,j,p,q} x^{j+q}.$$

Proof. The proof is similar to that of Lemma 3.16. Any element of $(\mathcal{K} \otimes_\pi \mathcal{T}) \otimes A$ can be written as $\omega = \sum_{t=1}^{M} m^{(t)} \otimes f^{(t)}$ where $m^{(t)} \in \mathcal{K} \otimes_\pi \mathcal{T}$. Therefore we have $m^{(t)} = \sum_{i,j \geq 0} e_{i,j} \otimes m_{i,j}^{(t)}$ where $m_{i,j}^{(t)} \in \mathcal{T}$ are rapidly decreasing. We can write each $m_{i,j}^{(t)}$ as

$$m_{i,j}^{(t)} = \sum_{p,q \geq 0} c_{i,j,p,q}^{(t)} S^p S^{*q}.$$

Therefore we have

$$\begin{aligned}
\omega &= \sum_{t=1}^{M} \left(\sum_{i,j \geq 0} e_{i,j} \otimes \sum_{p,q \geq 0} c_{i,j,p,q}^{(t)} S^p S^{*q} \right) \otimes f^{(t)} \\
&= \sum_{i,j,p,q \geq 0} e_{i,j} \otimes S^p S^{*q} \otimes \left(\sum_{t=1}^{M} c_{i,j,p,q}^{(t)} f^{(t)} \right)
\end{aligned}$$

Let $\omega = \sum_{i,j,p,q\geq 0}^{\infty} e_{i,j} \otimes S^p S^{*q} \otimes a_{i,j,p,q}$ be an element of $\bar{\mathcal{C}}$, that is $a_{i,j,p,q} \in A_{i+p}A_{-(j+q)}$. By Lemma 3.3, we can write

$$f^{(t)} = \sum_{k=0}^{N} P_k^{(t)}(h)y^k + \sum_{k=1}^{N} P_{-k}^{(t)}(h)x^k.$$

Let D be the maximum degree of the polynomials $P_k^{(t)}$ and n the degree of the polynomial P defining $A = \mathbb{C}[h](\sigma, P)$. Just like in the proof of Lemma 3.16, we have $a_{i,j,p,q} \neq 0$ only if $|i - j + p - q| \leq N$. We also have that $a_{i+p,-(j+q)} \neq 0$ implies $i + p \leq \frac{D}{n}$ or $j + q \leq \frac{D}{n}$. These conditions imply that $a_{i+p,-(j+q)} \neq 0$ only if $i + j + p + q \leq N + \frac{2D}{n}$ and thus only for finitely many i, j, p and q. □

Define $\eta : \mathcal{T}_A \to \bar{\mathcal{C}}$ as the restriction of the injective morphism $\mathcal{T} \otimes A \to (\mathcal{T} \otimes_\pi \mathcal{T}) \otimes A$ given by $\eta(x \otimes a) = e \otimes x \otimes a$.

Proposition 4.8. *There is a Morita equivalence between $\bar{\mathcal{C}}$ and $\eta(\mathcal{T}_A)$. Therefore $kk(\eta) \in kk_0^{\mathrm{alg}}(\mathcal{T}_A, \bar{\mathcal{C}})$ is invertible.*

Proof. Using Lemma 3.9, we have an injective morphism $\bar{\mathcal{C}} \to (\mathcal{T} \otimes_\pi \mathcal{T}) \otimes \mathcal{E}$ (where $\mathcal{E} = \mathcal{E}_1$ if $q = 1$ and $\mathcal{E} = \mathcal{E}_2$ if $q \neq 1$). The Morita equivalence is given by $\xi_i = \xi_i' = e_{i,0} \otimes 1 \otimes \mathcal{U}_1^i$ and $\eta_j = \eta_j' = e_{0,j} \otimes 1 \otimes \mathcal{U}_{-1}^j$. The proof is similar to the proof of Theorem 4.1. □

Theorem 4.9. *$kk(j_0) \in kk_0^{\mathrm{alg}}(A_0, \mathcal{T}_A)$ is invertible.*

Proof. By Proposition 4.5, we know that $kk(j_0)$ has a left inverse and $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$ has a right inverse. Now, we prove that $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$ has a left inverse, which completes the proof.

Since $\phi_0(S) = S \otimes 1$, then if $a_i \in A_i$ and $a_{-j} \in A_{-j}$, we have

$$\Phi_0(e_{i,j} \otimes a_i a_{-j}) = e_{i,j} \otimes 1 \otimes a_i a_{-j} \in \bar{\mathcal{C}}$$

and therefore $\Phi_0(\mathcal{C}) \subseteq \bar{\mathcal{C}}$, thus by item (4) of Proposition 2.28 we have

$$kk(\Phi_0|_{\mathcal{C}}) \circ kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) = kk(\Phi_0, \Phi_0 \circ \mathrm{Ad}(S \otimes 1)).$$

By item (5) of Proposition 2.28, we obtain

$$kk(\Phi_0, \Phi_0 \circ \mathrm{Ad}(S \otimes 1)) = kk(\Phi_1, \Phi_0 \circ \mathrm{Ad}(S \otimes 1)).$$

We have $\phi_1(S) = S^2S^* \otimes 1 + e \otimes S$ and therefore $\Phi_1 - \Phi_0 \circ \mathrm{Ad}(S \otimes 1) = \eta$. By item (2) of Proposition 2.28, $kk(\Phi_1, \Phi_0 \circ \mathrm{Ad}(S \otimes 1)) = kk(\eta)$ and by Lemma 4.8, $kk(\eta)$ is invertible. □

With the isomorphisms in $\mathfrak{KK}^{\mathrm{alg}}$ from Theorems 4.1 and 4.9, we construct the desired exact triangle.

Theorem 4.10. *For a generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P(h))$ with P a non-constant polynomial and*

- *$q = 1$ and $h_0 \neq 0$ or*
- *q is not a root of unity and P has a root different from $\frac{h_0}{1-q}$*

there is an exact triangle

$$SA \to A_1A_{-1} \xrightarrow{0} A_0 \to A.$$

Proof. The linearly split extension

$$0 \to \Lambda_A \xrightarrow{\iota} \mathcal{T}_A \xrightarrow{\bar{p}} A \to 0 \tag{4.1}$$

yields an exact triangle

$$SA \xrightarrow{kk(E)} \Lambda_A \xrightarrow{kk(\iota)} \mathcal{T}_A \xrightarrow{kk(\bar{p})} A,$$

where $kk(E) \in kk_1^{\mathrm{alg}}(A, \Lambda_A) \cong kk_0^{\mathrm{alg}}(SA, \Lambda_A)$ is the element defined by the extension (4.1).

By Theorem 4.1, the inclusion $j_1 : A_1A_{-1} \to \Lambda_A$ defined by $j_1(x) = e_{00} \otimes x$ induces an invertible element $kk(j_1) \in kk_0^{\mathrm{alg}}(A_1A_{-1}, \Lambda_A)$. By Theorem 4.9, the inclusion $j_0 : A_0 \to \mathcal{T}_A$ defined by $j_0(a) = 1 \otimes a$ induces an invertible element $kk(j_0) \in kk_0^{\mathrm{alg}}(A_0, \mathcal{T}_A)$. We define ϕ by the commutative diagram in $\mathfrak{KK}^{\mathrm{alg}}$

$$\begin{array}{ccc} \Lambda_A & \xrightarrow{kk(\iota)} & \mathcal{T}_A \\ {\scriptstyle kk(j_1)}\uparrow & & \downarrow{\scriptstyle kk(j_0)^{-1}} \\ A_1A_{-1} & \xrightarrow[\phi]{} & A_0 \end{array}$$

and claim that

$$\phi = kk(i) - kk(\sigma \circ i). \tag{4.2}$$

Here $i : A_1A_{-1} \to A_0$ is the inclusion and σ is the automorphism of $\mathbb{C}[h]$ defining $A = \mathbb{C}[h](\sigma, P)$. For this we use Proposition 4.5 to obtain

$$kk(j_0)^{-1} = \kappa \circ kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$$

and therefore

$$kk(j_0)^{-1} \circ kk(\iota) \circ kk(j_1) = \kappa \circ kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) \circ kk(\iota) \circ kk(j_1).$$

Let $x = R(h) \in A_1A_{-1} \subseteq \mathbb{C}[h]$. The composition $kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1)) \circ kk(\iota) \circ kk(j_1)$ corresponds to the quasihomomorphism $(\phi, \psi) : A_1A_{-1} \rightrightarrows \mathcal{T} \otimes A \triangleright \mathcal{C}$, where $\phi(x) = e_{00} \otimes x$ and $\psi(x) = e_{11} \otimes x$. Since ϕ and ψ are orthogonal, we obtain $kk(\phi, \psi) = kk(\phi) - kk(\psi)$. Now we compose this difference by the Morita equivalence κ of Proposition 4.4. Thus we have that $\kappa \circ kk(\phi)$ and $\kappa \circ kk(\psi)$ are determined by maps $A_1A_{-1} \to \mathcal{C} \to \mathcal{K} \otimes A_0$ that send $x \mapsto e_{00} \otimes x$ and $x \mapsto e_{00} \otimes \rho^{-1}(\mathcal{U}_{-1}R(G)\mathcal{U}_1) = e_{00} \otimes R(\sigma(h))$ (here we use the representation ρ of Lemma 3.9). Thus we conclude $\phi = kk(i) - kk(\sigma \circ i)$, proving (4.2).

Now we prove that $\phi = 0$. Both i and $\sigma \circ i$ factor through a contractible subalgebra of $\mathbb{C}[h]$. This is because we have $i(A_1A_{-1}) = P(h)\mathbb{C}[h]$ and $\sigma(A_1A_{-1}) = P(\sigma(h))\mathbb{C}[h]$ and the polynomials $P(h)$ and $P(\sigma(h))$ have some linear factors $L(h)$ and $L(\sigma(h))$. Thus the morphisms i and σ factor through the subalgebras $L(h)\mathbb{C}[h]$ and $L(\sigma(h))\mathbb{C}[h]$, which are contractible. Therefore we have $kk(i) = kk(\sigma \circ i) = 0$. □

Lemma 4.11. *Let $(\mathfrak{T}, \Sigma)$ be a triangulated category. If there is an exact triangle*

$$\Sigma X \to Y \xrightarrow{0} Z \to X$$

then $X \cong Z \oplus \Sigma^{-1}Y$.

Proof. See Corollary 1.2.7 in [18]. □

Now we compute the isomorphism class of A_1A_{-1} in $\mathfrak{KK}^{\mathrm{alg}}$.

Proposition 4.12. *Let $A = \mathbb{C}[h](\sigma, P)$ where P is a nonconstant polynomial with r different roots, then*

$$A_1A_{-1} \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C}^{r-1}.$$

Proof. Let $P(h) = c(h-h_1)^{n_1}\cdots(h-h_r)^{n_r}$. Without loss of generality we can assume $c = 1$. Since $A_1A_{-1} = (P(h))$ we have a linearly split extension

$$0 \to A_1A_{-1} \to \mathbb{C}[h] \xrightarrow{\pi} \mathbb{C}[h]/(P(h)) \to 0. \tag{4.3}$$

By the Chinese Remainder Theorem, there is an isomorphism

$$\phi : \mathbb{C}[h]/(P(h)) \to \prod_{i=1}^{r} \mathbb{C}[h]/(h-h_i)^{n_i}.$$

We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (h-h_i)^{n_i}\mathbb{C}[h] & \longrightarrow & \mathbb{C}[h] & \xrightarrow{q_i} & \mathbb{C}[h]/(h-h_i)^{n_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \mu_i} & & \\
0 & \longrightarrow & (h-h_i)\mathbb{C}[h] & \longrightarrow & \mathbb{C}[h] & \xrightarrow[\mathrm{ev}_{h_i}]{} & \mathbb{C} & \longrightarrow & 0
\end{array}$$

Since $(h-h_i)^{n_i}\mathbb{C}[h]$ and $(h-h_i)\mathbb{C}[h]$ are contractible, $kk(q_i)$ and $kk(\mathrm{ev}_{h_i})$ are invertible, therefore $kk(\mu_i) \in kk_0^{\mathrm{alg}}(\mathbb{C}[h]/(h-h_i)^{n_i}, \mathbb{C})$ is invertible. By the additivity of $\mathfrak{KK}^{\mathrm{alg}}$, the homomorphism $\mu : \prod_{i=1}^{r} \mathbb{C}[h]/(h-h_i)^{n_i} \to \mathbb{C}^r$ given by μ_i in the i-th component induces an invertible element $kk(\mu)$. Note that $\mu \circ \pi : \mathbb{C}[h] \to \mathbb{C}^r$ is given by ev_{h_i} in the i-th component.

Since all evaluation maps ev_{h_i} induce the same kk^{alg}-isomorphism $kk(\mathrm{ev}_0)$ in $kk^{\mathrm{alg}}(\mathbb{C}[h], \mathbb{C})$, we have the commutative diagram in $\mathfrak{KK}^{\mathrm{alg}}$

$$\begin{array}{ccc}
\mathbb{C}[h] & \xrightarrow{kk(\pi)} & \mathbb{C}[h]/P(h) \\
{\scriptstyle kk(\mathrm{ev}_0)} \downarrow & & \downarrow {\scriptstyle kk(\mu)} \\
\mathbb{C} & \xrightarrow[kk(\triangle)]{} & \mathbb{C}^r
\end{array}$$

where $\triangle : \mathbb{C} \to \mathbb{C}^r$ is the diagonal morphism $\triangle(1) = (1, \dots, 1)$. Replacing $\mathbb{C}[h]$ by $\mathbb{C}$ and $\mathbb{C}[h]/(P(h))$ by $\mathbb{C}^r$ in the exact triangle corresponding to (4.3), we obtain an exact triangle in $\mathfrak{KK}^{\mathrm{alg}}$

$$S\mathbb{C}^r \to A_1A_{-1} \to \mathbb{C} \overset{kk(\triangle)}{\to} \mathbb{C}^r. \tag{4.4}$$

The linearly split extension $0 \to \mathbb{C} \overset{\triangle}{\to} \mathbb{C}^r \to \mathbb{C}^{r-1} \to 0$ yields an exact triangle

$$S\mathbb{C}^{r-1} \to \mathbb{C} \overset{kk(\triangle)}{\to} \mathbb{C}^r \to \mathbb{C}^{r-1}.$$

Rotating this triangle we obtain the exact triangle

$$S\mathbb{C}^r \to S\mathbb{C}^{r-1} \to \mathbb{C} \overset{kk(\triangle)}{\to} \mathbb{C}^r. \tag{4.5}$$

Since both triangles (4.4) and (4.5) complete the morphism $kk(\triangle) : \mathbb{C} \to \mathbb{C}^r$, by the axiom TR3 of triangulated categories we have $A_1 A_{-1} \cong_{\mathfrak{KK}^{\mathrm{alg}}} S\mathbb{C}^{r-1}$. □

Theorem 4.13. *Let $A = \mathbb{C}[h](\sigma, P(h))$ be generalized Weyl algebra with $\sigma(h) = qh + h_0$ and P a non-constant polynomial such that*

- *$q = 1$ and $h_0 \neq 0$ or*
- *q is not a root of unity and P has a root different from $\frac{h_0}{1-q}$.*

Then $A \cong_{\mathfrak{KK}^{\mathrm{alg}}} \mathbb{C}^r$.

Proof. The result follows from Theorem 4.10, Lemma 4.11 and Proposition 4.12. □

Corollary 4.14. *Let A be as in Theorem 4.13. Then $A \cong \mathbb{C}^r$ in $\mathfrak{KK}^{\mathcal{L}_p}$ and so*

$$kk_0^{\mathcal{L}_p}(\mathbb{C}, A) = \mathbb{Z}^r \quad \textit{and} \quad kk_1^{\mathcal{L}_p}(\mathbb{C}, A) = 0.$$ □

Corollary 4.14 implies $K_0(A \otimes_\pi \mathcal{L}_p) = \mathbb{Z}^r$. This is compatible with Theorem 4.5 of [17], which computes $K_0(A) = \mathbb{Z}^r$ for $A = \mathbb{C}[h](\sigma, P)$ when $\sigma(h) = h - 1$ and P has r simple roots.

Examples 4.15. We apply Theorem 4.13 in the following cases.

1. The quantum Weyl algebra A_q with $q \neq 1$ not a root of unity is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$.

2. In the case of the primitive factors B_λ of $U(\mathfrak{sl}_2)$, we have $P(h) = -h(h+1) - \lambda/4$. If $\lambda = 1$, then $B_\lambda \cong \mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$. If $\lambda \neq 1$, then $B_\lambda \cong \mathbb{C}^2$ in $\mathfrak{KK}^{\mathrm{alg}}$. This implies $kk_0^{\mathcal{L}_p}(\mathbb{C}, B_\lambda) = \mathbb{Z} \oplus \mathbb{Z}$ and $kk_1^{\mathcal{L}_p}(\mathbb{C}, B_\lambda) = 0$.

3. The quantum weighted projective line $\mathcal{O}(\mathbb{WP}_q(k,l))$ is isomorphic to $\mathbb{C}[h](\sigma, P)$ with $\sigma(h) = q^{2l}h$ and
$$P(h) = h^k \prod_{i=0}^{l-1} (1 - q^{-2i}h).$$

In the case $q \neq 1$ is not a root of unity, we have $\mathcal{O}(\mathbb{WP}_q(k,l)) \cong \mathbb{C}^{l+1}$ in $\mathfrak{KK}^{\mathrm{alg}}$. This implies $kk_0^{\mathcal{L}_p}(\mathbb{C}, \mathcal{O}(\mathbb{WP}_q(k,l))) = \mathbb{Z}^{l+1}$ and $kk_1^{\mathcal{L}_p}(\mathbb{C}, \mathcal{O}(\mathbb{WP}_q(k,l))) = 0$. (Compare with Corollary 5.3 of [4].)

Now we treat the case the case where q is not a root of unity and P has only $\frac{h_0}{1-q}$ as a root. We will use the following lemma.

Lemma 4.16. *Let $A = \bigoplus_{n\in\mathbb{N}} A_n$ be an $\mathbb{N}$-graded locally convex algebra with the fine topology, then A is diffotopy equivalent to A_0. In particular $\mathbb{C}[h]$ is diffotopy equivalent to $\mathbb{C}$.*

Proof. The diffotopy is given by the family of morphisms $\phi_t : A \to A$, $t \in [0,1]$, sending an element $a_n \in A_n$ to $t^n a_n$. When $t = 1$ we recover the identity and when $t = 0$ the morphism is a retraction of A onto A_0. □

Proposition 4.17. *The generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P(h))$, with $\sigma(h) = qh + h_0$ such that $q \neq 1$ and P has only $\frac{h_0}{1-q}$ as a root, is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$.*

Proof. By Proposition 3.6, A is isomorphic to $\mathbb{C}[h](\sigma_1, P_1)$ with $\sigma_1(h) = qh$ and $P_1(h) = ch^n$ with $c \in \mathbb{C}^*$ and $n \geq 1$. The algebra $\mathbb{C}[h](\sigma_1, P_1)$ is $\mathbb{N}$-graded with $\deg h = 2$, $\deg x = n$ and $\deg y = n$. To prove this we check that the defining relations

$$xh = qhx,\ yh = q^{-1}hy,\ yx = ch^n \text{ and } xy = cq^n h^n$$

are compatible with the grading.

The result follows from applyng Lemma 4.16 again, since the degree 0 subalgebra of A is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$. □

Example 4.18. The quantum plane $\mathbb{C}[h](\sigma, h)$ with $\sigma(h) = qh$ is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$.

4.2 The case where P is a constant polynomial

If P is a nonzero constant polynomial and q, h_0 are real, then $A = \mathbb{C}[h](\sigma, P)$ is a tame smooth generalized crossed product and we can apply the results from [15]. If q or h_0 are not real, we can still obtain results similar to those of [15].

Proposition 4.19. *Let* $A = \mathbb{C}[h](\sigma, P)$ *where* $P \neq 0$ *is a constant polynomial, then* $A \cong S\mathbb{C} \times \mathbb{C}$ *in* $\mathfrak{KK}^{\mathrm{alg}}$. *This implies* $A \cong S\mathbb{C} \times \mathbb{C}$ *in* $\mathfrak{KK}^{\mathcal{L}_p}$ *and therefore we have* $kk_0^{\mathcal{L}_p}(\mathbb{C}, A) = \mathbb{Z}$ *and* $kk_1^{\mathcal{L}_p}(\mathbb{C}, A) = \mathbb{Z}$.

Proof. Even though A might not be a tame smooth generalized crossed products, it has a frame $\xi_i = y^i$ and $\bar{\xi}_i = x^i$ for $i \in \mathbb{N}$ that satisfies the conditions of Definition 18 in [15]. Following the proofs of sections 8 and 9 of [15] it can be shown that the linearly split extension

$$0 \to \Lambda_A \xrightarrow{\iota} \mathcal{T}_A \xrightarrow{\bar{p}} A \to 0,$$

yields an exact triangle

$$SA \xrightarrow{kk(E)} \Lambda_A \xrightarrow{kk(\iota)} \mathcal{T}_A \xrightarrow{kk(\bar{p})} A.$$

By Theorem 27 of [15], $j_1 \colon \mathbb{C}[h] \to \Lambda_A$, defined by $j_1(x) = e_{00} \otimes x$ induces an invertible element $kk(j_1)$ and by Theorem 33 of [15], $j_0 \colon \mathbb{C}[h] \to \mathcal{T}_A$ defined by $j_0(x) = 1 \otimes x$ induces an invertible element $kk(j_0)$. We have a commutative diagram in $\mathfrak{KK}^{\mathrm{alg}}$

$$\begin{array}{ccc} \Lambda_A & \xrightarrow{kk(\iota)} & \mathcal{T}_A \\ {\scriptstyle kk(j_1)} \uparrow & & \uparrow {\scriptstyle kk(j_0)} \\ \mathbb{C}[h] & \xrightarrow[\alpha]{} & \mathbb{C}[h]. \end{array}$$

We prove that $\alpha = 1_{\mathbb{C}[h]} - kk(\sigma)$ and that $1_{\mathbb{C}[h]} = kk(\sigma)$, thus concluding that $\alpha = 0$. By Theorem 33 of [15], we have $kk(j_0)^{-1} = kk(\iota_1)^{-1} \circ kk(\mathrm{id}, \mathrm{Ad}(S \otimes 1))$. The composition $kk(1, \mathrm{Ad}(S \otimes 1)) \circ kk(\iota) \circ kk(j_1)$ corresponds to a quasi-homomorphism

$$(\phi, \psi) \colon \mathbb{C}[h] \rightrightarrows \mathcal{T} \otimes A \triangleright \mathcal{C},$$

where $\phi(Q) = e_{00} \otimes Q$ and $\psi(Q) = e_{11} \otimes Q$ for all $Q \in \mathbb{C}[h]$. Since ϕ and ψ are orthogonal $kk(\phi, \psi) = kk(\phi) - kk(\psi)$. We now compose $kk(\phi)$ and $kk(\psi)$ with $kk(j_1)^{-1}$. Theorem 27 of [15] characterizes $kk(j_1)^{-1}$ as given by a Morita equivalence defined by

$$\Xi_i = S^i \otimes y^i \quad \text{and} \quad \overline{\Xi}_i = S^{*i} \otimes x^i.$$

therefore $kk(j_1)^{-1} \circ kk(\phi)$ is defined by the morphism $Q \mapsto Q$ and $kk(j_1)^{-1} \circ kk(\psi)$ is defined by $Q \mapsto xQy = \sigma(Q)$. This implies that $\alpha = 1_{\mathbb{C}[h]} - kk(\sigma)$.

The commutative diagram

$$\begin{array}{ccc} \mathbb{C}[h] & \xrightarrow[\sigma]{} & \mathbb{C}[h] \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & {\scriptstyle \mathrm{ev}_0}\downarrow \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}, \end{array}$$

implies that $kk(\sigma) = 1_{\mathbb{C}[h]}$ and thus $\alpha = 0$.

This implies the existence of an exact triangle in $\mathfrak{KK}^{\mathrm{alg}}$

$$SA \to \mathbb{C} \xrightarrow{0} \mathbb{C} \to A.$$

Using Lemma 4.11, we obtain $A \cong S\mathbb{C} \oplus \mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$. □

In the case where $P = 0$ we have the following result.

Proposition 4.20. *The generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P(h))$ with $P = 0$ is isomorphic to $\mathbb{C}$ in $\mathfrak{KK}^{\mathrm{alg}}$.*

Proof. The relations

$$xh = \sigma(h)x, \; yh = \sigma^{-1}(h)y, \; yx = 0 \text{ and } xy = 0$$

are compatible with the grading determined by $\deg h = 0$, $\deg x = 1$ and $\deg y = 1$, therefore the algebra A is $\mathbb{N}$-graded. The result follows from Lemma 4.16 and the fact that the degree 0 subalgebra of A is equal to $\mathbb{C}[h]$. □

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