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FACULTAD DE CIENCIAS

TESIS

“The Electric Charge in multiplets of Gauge Theories”

PARA OBTENER EL GRADO ACADÉMICO DE DOCTOR EN CIENCIAS CON MENCIÓN EN FÍSICA

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Introduction

The electric charge is a conserved quantity as far as we know. It is a property of matter which constitutes one of the most important milestones in Physics at the same level as conservation of energy. Its minimal value (without considering the quarks that cannot be isolated) has been experimentally determined as a quantized quantity since the nineteenth century and it is currently accepted as a constant (and exact value) of nature $e = 1.602\ 176\ 634 \times 10^{-19}$ C [1].

From a theoretical point of view, electric charge is recognized as the global conserved charge of the Quantum Electrodynamics (QED) symmetry group, $U(1)_Q$. In this context, it is a Noether's charge. However, at high energies QED is not the governing symmetry anymore, i.e. there is another gauge theory which explains the particle interactions above certain limit called electroweak energy scale: The Standard Model (SM). Thus, we have to understand how electric charges are predicted at this stage even though they are not defined at high energies; this is the main purpose of this work.

SM is a widely accepted theory thanks to its enormous success in predicting what would later be important discoveries such as the very existence of some particles, properties of other particles already discovered and outcomes of some particle physics experiments. This gauge theory belongs to the symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. For better understanding, it can be separated into (i) $SU(3)_C$, which corresponds to Quantum Chromodynamics (QCD), theory developed in the early 70's by Politzer [2], Wilczek and Gross [3] in order to describe strong interactions; and (ii) $SU(2)_L \otimes U(1)_Y$ that corresponds to Electroweak Standard Model (EWSM) developed in the middle 60's by Weinberg, Salam and Glashow. The latter describes the electromagnetic and weak interactions in the same framework at high energies. The EWSM prototype was done by Glashow [4], while Weinberg and Salam [5, 6] implemented the Higgs mechanism [7] to generate masses to all particles involved and succeeded in placing the model as a gauge field theory.

It is in this EWSM gauge group that some of the most important predictions were made such as the existence of the vector bosons W and Z as well as the scalar Higgs boson, all of them confirmed experimentally some years later.

All interactions are written to be a local gauge invariant in the Lagrangian density, which means that the unitary gauge transformations of all the fields are defined for that purpose. Also, within this electroweak Lagrangian there are all the interactions between the fields and their respective gauge bosons that conveying the forces. These gauge bosons (W_1^μ, W_2^μ, W_3^μ and B^μ for $SU(2)_L \otimes U(1)_Y$) are massless until a process called Spontaneous Symmetry Breaking (SSB) takes place. In fact, all particles acquire mass after this process.

Despite all the virtues of SM, it must be said that there are some reasons to believe that it is an incomplete theory and should be considered, at best, as an effective theory that describes particle physics only on energy scales that have been experimentally tested; that is, up to about 13 TeV c.m. in Run 2 of the LHC[1].

Some of the items and questions that reveal the incompleteness of SM are the massive neutrinos, dark matter, matter-antimatter asymmetry, gravitational interaction, etc. and as a consequence, SM has to be extended to a more complete theory. In the search for these extensions it must be considered what types of new very massive particles could exist or what implications these could have on experiments at accessible energies.

EWSM extensions are described by left-right symmetry groups like $SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ or new chiral groups like $SU(3)_L \otimes U(1)_N$. The symmetries inherent to these new theories occur at high energies and the massless particles represented as new multiplets will be disjointed to generate massive physical fields when the symmetry breaks down to $U(1)_Q$, at energies in which experiments are developed.

That means that when energy decreases, a limit is evidenced when the symmetry is broken in a process called "Spontaneous Symmetry Breaking" (SSB) as a result of Higgs mechanism. For example, above the electroweak scale in EWSM ($100 \sim 1000$ GeV), Lagrangian obeys the symmetry $SU(2)_L \otimes U(1)_Y$ where there are no masses at all and the hypercharge is the respective conserved charge[2]. After this breakdown the symmetry is that of the group $U(1)_Q$ where the electric charge is conserved. In the “breaking” process the particles acquire mass so that the number of degrees of freedom remains unchanged, according to the Goldstone theorem [8].

[1]The design energy is 14 TeV c.m. to be reached in run 3

[2]As a consequence of Noether’s theorem, by means of it all continuous symmetry leads to a conserved current and charge.

This work demonstrates that electric charge of all particles can be predicted from the very beginning, when the EWSM Lagrangian is defined, and we show how to do it from different symmetry groups. We are going to assign electric charges in gauge theories such as the minimal left-right theory that considers the conservation of parity from the beginning and belongs to the symmetry $SU(2)_L \otimes SU(2)_R \otimes U(1)_Y$; the 331 theories which explain why there are three fermionic families and belongs to $SU(3)_L \otimes U(1)_N$. Moreover, if we require a gauge theory that includes the previous two, we must take into account the model $SU(3)_R \otimes U(1)_X$ [9–12] and this is also analyzed. All these symmetries are built respecting the conservation of electric charge after the breakdown and this fact is used to elaborate charge operators for the different multiplets. In order to do that, we describe how the SSB occurs also in the gauge transformation operators getting the electric charges and how they are obtained for almost all multiplets involved in each symmetry group studied here. In order to achieve that, we have to mention that all of the mathematical deductions are formally discussed using classical unitary transformations for gauge theories as is customary in literature.

One of the motivations for this work is that in publications where SM and extensions are discussed, it is not usual to read a description of the criteria used to choose the particle content inside the multiplets, presenting them a priori with their respective electric charges. Here, we suggest that the very process of obtaining electric charge is of the utmost importance in the multiplets construction at high energies (when breaking is not carried out yet) and on the other hand, charge operators contextualize the symmetry breaking process within the principle of electric charge conservation.

To achieve this, the gauge transformation operators must be set up for each multiplet of the theory depending on the type of multiplet we are considering. So, the relation among the conserved charges of the symmetries involved in the breaking the electric charges are obtained is an application of the Gell-Mann Nishijima [13] formula applied to the electroweak interaction whose mathematical structure is analyzed in the context of a gauge theory. However, this formula is useful for column vector representations but it is not very clear for other types of representations. This work is focused on the latter.

Then, the aim of this thesis is to achieve the electric charge operator for exotic multiplets in different gauge symmetries and demonstrate that it is not necessary to build them up from the subsequent interactions after the SSB.

Therefore, the charge operators are the main issues in this work and its concept has to be examined. Since gauge theories are associated with local symmetries and they are generalized from global ones, they posses a conserved physical charge. These charges appear to be generators of their respective symmetry groups. In this way, electric charges are generators of the global $U(1)_Q$ symmetry group included in the QED. Besides, weak isospin and hypercharge constitute three generators of the $SU(2)_L \otimes U(1)_Y$ group of the electroweak theory in the Standard Model, or the nine generators of $SU(3)_L \otimes U(1)_N$.

These conserved quantities are eigenvalues of Hermitian operators that commute with the Hamiltonian. For example, electric charges are eigenvalues of Q operator which commutes with Hamiltonian[3] $[Q, H]$, and its expected values q are constants of the theory. Applied to a field particle $Q\psi = q\psi$, where q is the expected value of Q and ψ is its eigenstate.

On the other hand, in the quantization process of a field theory satisfying the unitarity conditions, it has demonstrated, see ref. [14], that charge operator must commute with linear and angular momentum operators. Also, the electric charge of a particle and its respective antiparticle have opposite signs, so that Q gets the total charge, $Q = q\left(N_{\psi} - N_{\bar{\psi}}\right)$, where N gives the particle and/or antiparticle count as seen in appendix (C).

The charge operator varies depending on the multiplet representation to which it is applied. In this work Q is calculated for all multiples involved in the Standard Model (Chapter 1) as well as in three of its extensions (Chapters 2, 3 and 4).

[3]According with Ehrenfest's theorem

Chapter 1

Symmetry of Electroweak Standard Model

1.1 Description of Standard Model

The Standard Model (SM) of particle physics is treated as a gauge theory that describes the characteristics and properties of elementary particles as well as their interactions.[1]

These particles can be classified, according to their spin, into two groups: fermions and bosons.

a) **Fermions**: Half-integer spin particles distributed according to the Fermi-Dirac statistic and obey the Pauli exclusion principle. The SM classifies this type of particles according to the interactions that take place between them[1] in:

- **Leptons**: they do not experience strong interaction but may experience weak and/or electromagnetic.

- **Quarks**: They do experience strong interaction as well as weak and electromagnetic. Furthermore, they combine to form hadrons.

b) **Bosons**: Integer spin particles that do not obey the Pauli exclusion principle and follow

[1]SM describes three of the four fundamental interactions of nature: the Strong, the Weak and the Electromagnetic. Gravitational is not within the theory "yet".

the Bose-Einstein statistic. The SM classifies them into two groups:

- **Scalars**: They have null spin and are quanta of scalar fields (Lorentz invariants). In the SM there is only one: the Higgs boson, responsible for the masses of all the particles in the model.

- **Vectors**: They have spin one and are quanta of vector fields. Also called gauge bosons and are responsible for the interactions.

As seen in figure 1.1, SM classifies quarks and leptons in three generations or families that differ from each other by the order of magnitude among their masses. This classification is done even before the symmetry breakdown, when they are in doublet representations considering the masses they will acquire. It is also seen the four gauge bosons responsible for electroweak interactions and the scalar Higgs responsible for the mass acquisition in the SSB process. Physical states of weak vector bosons are revealed only after the breaking since before that they are part of covariant derivatives and are not physical eigenstates until mix each other appearing as the known $W^{\pm}$, Z and γ (photon) shown in the figure. With respect to the gluons, they belong to the colour symmetry which is not part of the symmetry breaking process. The Higgs is the lower (neutral) component of its former doublet.

Notice that there are 17 different elementary particles (in addition to the antiparticles that do not appear explicitly). Each of them is represented by a field within the Lagrangian, where all the interactions allowed by the theory are also included.

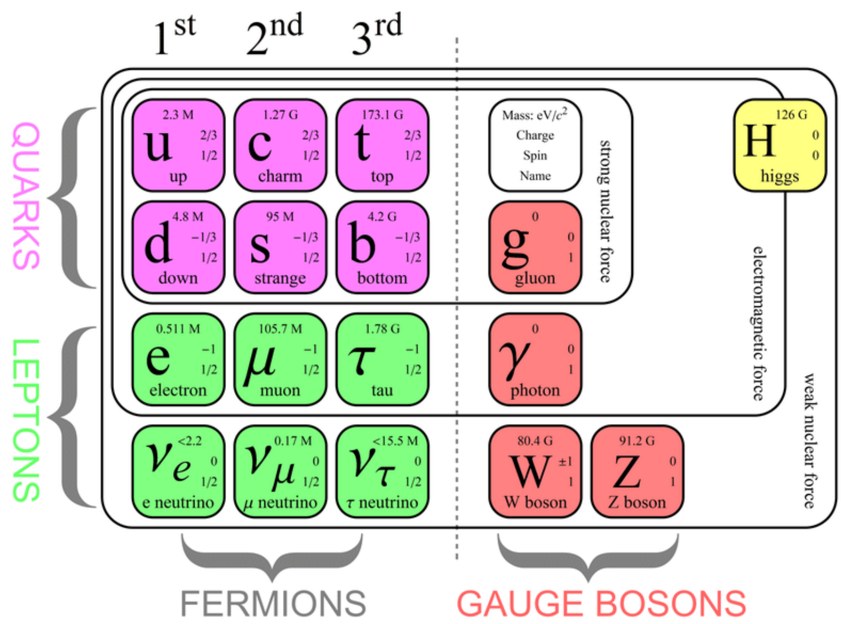

Figure 1.1: Standard Model. Source: PhD Thesis [15]

The reason why exactly three flavor families exist so far is unknown in SM domain; but there are some hypotheses in other areas, such as the anthropic explanation [16] or the cancellation of anomalies in higher symmetries [17].

With respect to the SM predictions, table 1.1 shows some of them and the year of experimental confirmation.

The particle fields have representations in $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry group, where C: Color, L: Left e Y: Hypercharge.

The interaction associated with $SU(3)_C$ is the strong force described by Quantum Chromodynamics (QCD), and the interaction associated with $SU(2)_L \otimes U(1)_Y$ is the electroweak force described as a weak hypercharge symmetry (for related papers in a historical context, see reference [18]). This last force is the unification, at high energies, of the weak and electromagnetic forces that manifest as such at low energies after a process called "Spontaneous Symmetry Breaking" (SSB).

On the other hand, SM is a chiral theory; this means that the left and right chiral projection fields are not transformed in the same way, the left fermions have doublet representation and right ones are singlet of the same symmetry group.

1.2 Standard Model extensions

The existence of deficiencies in the SM, such as those mentioned in the introduction, make it necessary to have a broader gauge theory that encompasses new particles and their consequent interactions.

The SM extensions are still under development, the neutrino mass is already taken into account within them and was not considered in the original model. There are a large number of extensions such as Left-Right symmetric models, large unification models, those that consider dark matter, etc. which try to improve what the SM cannot explain.

In the same symmetry group of the SM, we have for example simple extensions: Two-Higgs-doublet model (2HDM) [19], which has two Higgs doublets in its minimum version [20]. There are also other 2HDM including Axion models [21], Dark matter and neutrino masses [22]; etc.

Particle	Predicted/ Discovered	Spin number	Electric charge (e)	Color	Mass (MeV/c^2)
u	1964 1968	1/2	+2/3	r, g, b	$2{,}16^{+0{,}49}_{-0{,}26}$
d	1964 1968	1/2	-1/3	r, g, b	$4{,}67^{+0{,}48}_{-0{,}17}$
c	1970 1974	1/2	+2/3	r, g, b	$1{,}27 \pm 0{,}02^{\dagger}$
s	1964 1968	1/2	-1/3	r, g, b	96^{+11}_{-5}
t	1973 1995	1/2	+2/3	r, g, b	$172{,}9 \pm 0{,}4^{\dagger}$
b	1973 1977	1/2	-1/3	r, g, b	$4{,}18^{+0{,}03\dagger}_{-0{,}02}$
e	1874 1897	1/2	-1	none	$\sim 0{,}511$
μ	- 1936	1/2	-1	none	$\sim 105{,}658$
τ	1971 1975	1/2	-1	none	$1776{,}86 \pm 0{,}12$
ν_e	1930 1956	1/2	0	none	< 2 eV
ν_μ	1940s 1962	1/2	0	none	< 0,19 eV
ν_τ	1970s 2000	1/2	0	none	< 18,2 eV
g	1962 1978	1	0	8 colors	0^*
γ	- 1899	1	0	none	~ 0
W	1968 1983	1	± 1	none	$80{,}385 \pm 0{,}012^{\dagger}$
Z	1968 1983	1	0	none	$91{,}1876 \pm 0{,}0021^{\dagger}$
H	1964 2012	0	0	none	$125{,}10 \pm 0{,}14^{\dagger}$

Table 1.1: SM particles with mass values from PDG 2020 [1]. *Theoretical value; †in GeV

Beside this, there are also extensions that include a scalar triplet, getting other ways to set dark matter [23] and neutrino mass terms [24] into the Lagrangian.

In the next section we explore exotic multiplets included in some extensions, in addition to the known ones, and find the electric charge operators regardless of their interactions.

1.3 Standard model Lagrangian

As in all gauge theory, the Lagrangian describes characteristics of the symmetry in which the theory develops; some of these characteristics are its covariant derivatives, the representation of its multiplets and the types of couplings between the fields. In the minimum SM above the electroweak scale, left chiral fermions and scalars transform as doublets, while right chiral fermions as singlets.

$$
\begin{aligned}
\mathscr{L} \quad &= -\tfrac{1}{4}F_{i\mu\nu}F_i^{\mu\nu} - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} && \text{...Vector sector} \\
&+i\overline{\Psi}_{iL}\not{D}\Psi_{iL} + i\overline{\Psi}_{iR}\not{D}\Psi_{iR} + h.c. && \text{...Fermion- Vector sector} \\
&+\overline{\Psi}_{iL}y_{ij}\Psi_{jR}\Phi + h.c. && \text{...Interaction sector (Yukawa)} \\
&+\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right) - V(\Phi) && \text{...Scalar-Vector sector}
\end{aligned}
\tag{1.1}
$$

We have to remark that de covariant derivative for left fields are different from right fields as we will describe in the next chapters. Besides, when these derivatives arises, there will always be interactions with gauge fields. In consequence, in the fermion-vector sector, there are interactions between the fermions themselves and with vector fields after the symmetry breaking. The vector sector defines the interactions among vector bosons after breaking, which mediate the electroweak interactions. The tensor $F_{i\mu\nu}$ is a QED-like electromagnetic tensor. This sector accounts for the derivatives and interactions of $SU(2)_L$ gauge bosons and $G_{\mu\nu}$ is for the $U(1)_Y$ gauge boson. The second line describes the fermions and their respective gauge interactions, where Ψ_L's are lepton and quark doublets and Ψ_R are the respective lepton and quark singlets. The third line describes the interactions between fermions and scalar bosons represented as a doublet Φ from which the fermionic mass terms will be obtained. The last line shows both scalar-scalar and scalar-vector interactions. As in the fermionic case, these interactions will give rise to the mass terms of the scalar and vector bosons after the SSB.

The Lagrangian shows fields (1.1) in $SU(2)_L \otimes U(1)_Y$ representations where the hypercharge is conserved at high energies (above the electroweak scale). At low energies, the symmetry breaks spontaneously to $U(1)_Q$, where the electric charge is conserved. In this symmetry breaking process, the particles acquire mass and electric charge through the Higgs mechanism [7] after which the interactions can be verified in experiments.

For instance, one of the terms that result from expanding the fermionic sector (1.1) after the symmetry breaking is $-\frac{g}{2\sqrt{2}}\left[\overline{\nu}_e\gamma^\alpha\left(1-\gamma^5\right)e\right]W_\alpha^- + hc$, which corresponds to one decay mode of the bosons W.

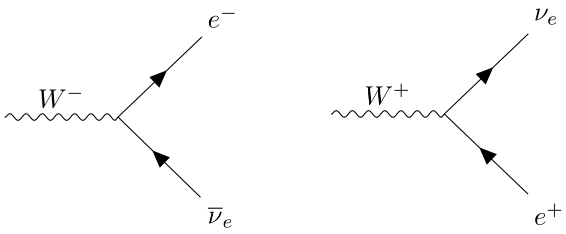

Figure 1.2: W boson decay

The decay rate is predicted by SM (at a tree level),

$$\Gamma\left(W \to e\nu\right) = \frac{G_F m_W^3}{6\pi\sqrt{2}} \approx 0,226 \text{ GeV} \tag{1.2}$$

Experimentally [1] the value is 0,223 GeV. This difference is due to loop calculations which did not take into account.

As well as this result, the SM predicts all the interactions between leptons, bosons and quarks, many of which have been experimentally validated and there is, as yet, no result that contradicts these predictions in a meaningful way.

1.4 Conserved charges

In the SM electroweak symmetry (EWSM), the conserved charge is the weak hypercharge, Y. After the SSB is carried out, the symmetry group changes from $SU(2)_L \otimes U(1)_Y$ to $U(1)_Q$ where the electric charge is the final conserved charge.

The electric charge operator is obtained as an extension of the original Gell-Mann Nishijima (GN) [25] formula adapted to the electroweak case,

$$Q = e\left(T_3 + \frac{Y}{2}\mathbb{1}_2\right). \tag{1.3}$$

T_{3L} is the $SU(2)_L$ third generator and Y (hypercharge) is the $U(1)_Y$ generator.

This formula relates conserved charges of the symmetries involved before and after the SSB. It will be clarified from the requirement of the vacuum expectation value invariance in next section.

1.4.1 $\mathrm{SU(2)_L \otimes U(1)_Y}$ Doublets

In this symmetry, left-handed fermions and scalars transform like doublets and can be represented as

$$\mathcal{D} = \begin{pmatrix} a \\ b \end{pmatrix}_L \sim (\mathbf{2}, Y)\,, \tag{1.4}$$

here **2** denotes $SU(2)_L$ doublet, in this case a "left doublet". The number Y is its respective $U(1)_Y$ hypercharge.

The predominant interaction is the electroweak force and the (unitary) transformations of these doublets are defined as

$$\begin{aligned} U_L U_Y \mathcal{D} &= e^{iT_j\alpha_j(x)} e^{i\frac{Y}{2}f(x)}\mathcal{D} \\ &\approx (\mathbb{1}_2 + iT_j\alpha_j(x))\left(1 + i\frac{Y}{2}f(x)\right)\mathcal{D} \\ &\approx \left(\mathbb{1}_2 + i\left(T_j\alpha_j(x) + \frac{Y}{2}f(x)\mathbb{1}_2\right)\right)\mathcal{D}, \end{aligned} \tag{1.5}$$

at first order. The matrices T_j, $j = 1, 2, 3$ are the three generators of $SU(2)_L$. This infinitesimal gauge transformation should be affected by the change in symmetry where the electric charge is the conserved charge and relates to hypercharge via equation (1.3).

To visualize how equation (1.5) changes, we have to point out that in the minimal EWSM with just one scalar doublet Φ, the symmetry $SU(2)_L \otimes U(1)_Y$ is broken to $U(1)_Q$ through the Higgs mechanism. After this process, the entire Lagrangian becomes $U(1)_Q$ invariant, as well as the vacuum expectation value (vev). The simplest way to do that is introducing a weak isospin scalar doublet $\Phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$ with degenerated minimum $\Phi_0 = \begin{pmatrix} \phi_0^a \\ \phi_0^b \end{pmatrix}$, $\left|\phi_a^0\right|^2 + \left|\phi_b^0\right|^2 \equiv v^2$. Then, a particular value is chosen for the ground state

$$\langle\Phi_0\rangle = \begin{pmatrix} \phi_a^0 \\ \phi_a^0 \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \; v \in \mathbb{R}.$$

Like in (1.5) scalar doublets transform as

$$\Phi \to \Phi' = U_L U_Y \Phi = \left(\mathbb{1}_2 + iT_j\alpha_j(x) + i\frac{Y}{2}f(x)\mathbb{1}_2\right)\Phi = \Phi + \delta\Phi\,. \tag{1.6}$$

Assuming the invariance of $\langle\Phi_0\rangle$,

$$\begin{aligned}\delta\langle\Phi_0\rangle &\approx i\left(T_j\alpha_j(x) + \frac{Y}{2}f(x)\mathbb{1}_2\right)\begin{pmatrix}0\\ v\end{pmatrix}\\ &= \frac{iv}{2}\begin{pmatrix}\alpha_1(x) - i\alpha_2(x)\\ -\alpha_3(x) + f(x)Y\end{pmatrix} \to \begin{pmatrix}0\\ 0\end{pmatrix},\end{aligned} \tag{1.7}$$

from which it follows that $\alpha_1(x) = \alpha_2(x) = 0$ and $\alpha_3(x) = f(x)Y$. In particular, to be in accordance with (1.3) and the fact that the last symmetry must be global, $Y(\Phi) = +1$ and $\alpha_3(x) = f(x) \to e$.

Then, the transformation (1.6) remains as

$$\begin{aligned}\delta\Phi &\approx ie\left(T_3 + \frac{1}{2}\mathbb{1}_2\right)\Phi\\ &= ieQ\Phi,\end{aligned} \tag{1.8}$$

taking into account that the electric charge is the conserved charge of the symmetry $U(1)_Q$ and $Q = T_3 + \frac{Y(\Phi)}{2}\mathbb{1}_2$. It should be emphasize that the value of the hypercharge, for the scalar case, is a consequence of the breaking process.

Therefore, after the breaking $U(1)_Q$ gauge transformation on the doublet scalar Φ using $Y(\Phi) = 1$ is

$$\delta\Phi \approx ie\left(T_3 + \frac{1}{2}\mathbb{1}_2\right)\Phi \approx i\begin{pmatrix}+e & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi^a\\ \phi^b\end{pmatrix}, \tag{1.9}$$

which tells us that $\delta\phi^a \to +ie\phi^a$ has an electric charge $+e$ and $\delta\phi^b \to 0\phi^b$ has no electric charge. Namely,

$$\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \tag{1.10}$$

Hence, in equation (1.5),

$$\delta\mathcal{D} \approx ie\left(T_3 + \frac{Y}{2}\mathbb{1}_2\right)\mathcal{D} = iQ\mathcal{D}. \tag{1.11}$$

Here, the symmetry breaking is evident when $U_L U_Y \to U_Q = e^{ieQ} \approx 1 + ieQ$. Then, the charge operator for doublets in SM is

$$Q(\mathcal{D}) = \frac{1}{2}\begin{pmatrix} 1+Y & 0 \\ 0 & -1+Y \end{pmatrix}, \tag{1.12}$$

in units of the positron charge e, which will be the units of Q used from now on.

In the minimal EWSM, left-handed fermions as well as scalars are in doublet representations. Each quark doublet $\mathcal{Q}_{iL} = \begin{pmatrix} u_{iL} & d_{iL} \end{pmatrix}^T$ is assumed to have a hypercharge[2] $Y(\mathcal{Q}_{iL}) = \frac{1}{3}$, so as

$$Q(\mathcal{Q}_{iL}) = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

is the electric charge operator which contains the charge eigenvalues for the known $u-$ and $d-$type quarks.

For the left-handed lepton case $L_i = \begin{pmatrix} \nu_{iL} & e_{iL} \end{pmatrix}^T$, it is assumed $Y(L_i) = -1$, resulting

$$Q(L_i) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

and it's consistent with the known electric charges too.

As already deduced previously, the scalar doublet has hypercharge $Y(\Phi) = +1$ and its electric charge operator is

$$Q(\Phi) = \begin{pmatrix} +1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is worth mentioning that an usual requirement for the electric charge conservation in the Lagrangian after breakdown is that the Higgs boson must be neutral. This is because in the vacuum, there are not charges at all and the Higgs doublet is arranged in such a way that it can couple with ad-hoc fields respecting the electric charge conservation. That is to say that in the previous result, the lower component of the doublet must be the field that gives rise to the Higgs boson and the upper component should be positive. However, there are currently models that propose to find positive and negatively charged Higgs bosons [26].

In short for fermions,

[2]Actually, the hypercharge value is taken so that the electric charge coincides with the known values.

1st Family	2nd Family	3rd Family	Isospin T_3	Charge Q	Hypercharge Y
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}_L$	$\begin{pmatrix} +\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}_L$	$+\frac{1}{3}$
u_R	c_R	t_R	0	$+\frac{2}{3}$	$+\frac{4}{3}$
d_R	s_R	b_R	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}_L$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}_L$	-1
ν_{eR}	$\nu_{\mu R}$	$\nu_{\tau R}$	0	0	0
e_R	μ_R	τ_R	0	-1	-2

Table 1.2: Charges for each fermionic family

1.4.2 $SU(2)_L \otimes U(1)_Y$ Singlets

In SM, right-handed fermions have a singlet representation and do not transform with the $SU(2)_L$ group. Then, the parameters must be $\alpha_j(x) = 0$ in (1.5) and the transformation is

$$U_Y \mathcal{S} \approx \left(1 + ie\frac{Y}{2}\right) \mathcal{S} \approx e^{iQ}\mathcal{S}, \tag{1.13}$$

so that the charge operator is $Q(\mathcal{S}) = \frac{Y}{2}$, which means that the electric charge is the hypercharge generator or $U_Y \to U_Q$.

In that way, we choose $Y(u_R) = \frac{4}{3}$ and $Y(d_R) = -\frac{2}{3}$ in order to obtain the known values $Q(u) = +\frac{2}{3}$ and $Q(d) = -\frac{1}{3}$.

Furthermore, hypercharges for right-handed leptons are $Y(\nu_{iR}) = 0$ and $Y(e_{iR}) = -2$ to get $Q(\nu) = 0$ and $Q(e) = -1$.

1.4.3 $SU(2)_L \otimes U(1)_Y$ Triplets

In SM, triplets are represented in the adjoint representation of $SU(2)$, i.e. they are grouped in 2×2 matrices, which have the appropriate dimensions to be coupled with doublets. Here, the equation (1.3) is not enough for giving electric charges because of it is defined for fundamental representations (column vector) and a new charge operator has to be performed.

It is shown in Appendix B that a vector-like triplet representation $\mathcal{T} = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix}^T$ that transforms with the $SU(2)$ adjoint matrices directly $\mathcal{T} \to \mathcal{T}' = \exp[i\theta_j \mathbb{T}_j]\mathcal{T}$, being $\mathbb{T}_j$ the adjoint

$SU(2)$ matrices (for spin 1), gives the same component transformations if we transform the 2×2 matrix $\widetilde{\mathcal{T}} \equiv \sigma_j \mathcal{T}$ as $\widetilde{\mathcal{T}}' = U_L \widetilde{\mathcal{T}} U_L^\dagger$, with $U_L = \exp\{i\theta_j T_j\}$, with T_j as the usual $SU(2)$ generators. For that reason, it is customary to say that $\widetilde{\mathcal{T}}$ is the adjoint representation.

Gauge (vector) bosons fall into this representation. In $SU(2)_L \otimes U(1)_Y$, the three gauge bosons corresponding to $SU(2)_L$ appear in $\widetilde{W}_\mu = \frac{\sigma_j}{2} W_{\mu j}$, $j = 1, 2, 3$ represented as

$$\widetilde{W}_\mu = \frac{1}{2} \begin{pmatrix} W_{3\mu} & W_{1\mu} - iW_{2\mu} \\ W_{1\mu} + iW_{2\mu} & -W_{3\mu} \end{pmatrix}. \tag{1.14}$$

It is known that gauge invariance for the $SU(N)_L$ group in covariant derivatives, requires $\widetilde{W}_\mu$ to transform as

$$\begin{aligned} \widetilde{W}_\mu \to \widetilde{W}'_\mu &= U_L \widetilde{W}_\mu U_L^\dagger + \frac{i}{g} \left(\partial_\mu U_L\right) U_L^\dagger \\ &\approx \left(\mathbb{1}_2 + i\alpha_j T_{jL}\right) \widetilde{W}_\mu \left(\mathbb{1}_2 - i\alpha_j T_{jL}^*\right) - \frac{1}{g} \left(\partial_\mu(\alpha_j T_{jL})\right) \left(\mathbb{1}_2 - i\alpha_j T_{jL}^*\right). \end{aligned} \tag{1.15}$$

As mentioned before, in the symmetry breaking $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = e$. Then,

$$\widetilde{W}_\mu \to \widetilde{W}'_\mu \approx \widetilde{W}_\mu + ie\Big[T_{3L}, \widetilde{W}_\mu\Big] = \widetilde{W}_\mu + ieQ\widetilde{W}_\mu.$$

So,

$$\delta\widetilde{W}_\mu = ieQ\widetilde{W}_\mu = ie\Big[T_3, \widetilde{W}_\mu\Big]. \tag{1.16}$$

The charge operator acts over the vector boson matrix as a commutator

$$Q(\widetilde{W}_\mu) \approx \Big[T_3, \widetilde{W}_\mu\Big] = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{1.17}$$

This matrix may be compared with (1.14) and it shows that the electric charge of physical $W_\mu^- \equiv \dfrac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}$ is $q(W_\mu^-) = -1$ and that of $W_\mu^+ \equiv \dfrac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}$ is $q(W_\mu^+) = +1$. Moreover, after the symmetry breaking, the neutral physical fields A_μ and Z_μ are linear combinations of real gauge fields W_μ^3 and B_μ, which is in accordance with the zero-valued eigenvalues of the last matrix diagonal and the fact that the term that includes B_μ would also be added in the diagonal.

Another way to see this is recalling that to the fourth gauge boson B_μ corresponds the $U(1)_Y$ generator, $\frac{Y}{2}$. This field transforms as $B_\mu \to B'_\mu = B_\mu - \partial_\mu f(x)$ and it is not affected by the $SU(2)_L$ transformations. Moreover, equation (1.17) does not change if B_μ is included. Let $\widetilde{V}_\mu \equiv gT_{3L}W_{\mu j} + g'\frac{Y}{2}B_\mu$, then it can be demonstrated that $\widetilde{V}'_\mu \approx \widetilde{V}_\mu + ie\Big[T_{3L} + \frac{Y}{2}, \widetilde{V}_\mu\Big]$, then

$$Q(\widetilde{V}_\mu) \approx \left[T_{3L} + \frac{Y}{2}\mathbb{1}, \widetilde{V}_\mu\right] = \left[T_{3L}, \widetilde{V}_\mu\right] \tag{1.18}$$

is the same result as (1.17). That means that including the B_μ bosons, Z_μ and A_μ have no electric charge as linear combinations of $W_{3\mu}$ and B_μ which are located in the diagonal of (1.17).

1.4.4 Exotic multiplets

In some SM extensions with the same symmetry group, scalar triplets are considered in the theory. For instance some scalar triplets belong to the $SU(2)$ adjoint representation like gauge vectors, which means that scalar or vector-matter fields are treated in the same way but with ad-hoc hypercharges. These triplets may predict neutrino masses [27] or the existence of Dark Matter (DM) [28].

Moreover, in some SM extension [29], a complex scalar triplet with $Y = 2$ has to be considered in order to perform the type-II seesaw model.

In general, the triplet $H = \begin{pmatrix} H_1 & H_2 & H_3 \end{pmatrix}^T \sim (\mathbf{3}_L, Y)$ can be written in its adjoint representation $\widetilde{H} = H_j\sigma_j = \begin{pmatrix} H_1 & h_2 \\ h_3 & -H_1 \end{pmatrix}$ but unlike the vector boson case, h_2 and h_3 are not conjugated, since H entries are not reals then $h_2 \equiv \frac{H_1 - iH_2}{\sqrt{2}} \neq h_3^* \equiv \frac{H_1^* - iH_2^*}{\sqrt{2}}$.

$$\begin{aligned}
\widetilde{H}' &= U_Y U_L \widetilde{H} U_L^\dagger \\
&\approx \left(\mathbb{1}_2 + ieT_{3L} + ie\frac{Y}{2}\mathbb{1}_2\right)\widetilde{H}\left(\mathbb{1}_2 - ieT_{3L}\right) \\
&\approx \widetilde{H} + ie\left(\left[T_{3L}, \widetilde{H}\right] + \frac{Y}{2}\widetilde{H}\right).
\end{aligned} \tag{1.19}$$

The charge operator is

$$Q\widetilde{H} = \left[T_3, \widetilde{H}\right] + \frac{Y}{2}\widetilde{H} = \begin{pmatrix} \frac{Y}{2} & 1 + \frac{Y}{2} \\ -1 + \frac{Y}{2} & \frac{Y}{2} \end{pmatrix}. \tag{1.20}$$

An appropriate term to generate a Majorana mass to neutrinos is $\overline{L^c}\epsilon\widetilde{H}L$ with $\epsilon = i\sigma_2$ [29].

This needs $Y(H) = +2$ because of $Y(L) = -1$. $\widetilde{H}$ has the following electric charge operator

$$Q(\widetilde{H}) = \begin{pmatrix} +1 & +2 \\ 0 & +1 \end{pmatrix}. \tag{1.21}$$

On the other hand, it could be generalized theories in the same symmetry group as SM with a generalized GN formula,

$$\frac{Q}{e} = \alpha T_3 + \frac{Y}{2}\mathbb{1}_2. \tag{1.22}$$

In this case, doublets have the following electric charge eigenvalues $Q(\mathcal{D}) = \frac{1}{2}\text{diag}(\alpha + Y, -\alpha + Y)$, triplets have $Q(\mathcal{T}) = \frac{1}{2}\text{diag}(2\alpha + Y, Y, -2\alpha + Y)$ and even quartets $Q(\mathcal{F}) = \frac{1}{2}\text{diag}(3\alpha + Y, \alpha + Y, -\alpha + Y, -3\alpha + Y)$, using (1.22) with appropriate dimensions for T_3. However, theories with $\alpha \neq 1$ do not include to SM in a way in which the doublets have the known charges of leptons and quarks.

One goal of this work is to construct multiplets with known electric charges without taking into account its interactions with other fields after SSB.

The cases of triplets and quartets are useful to illustrate the method exposed in this thesis. The triplets are built from two doublets and may be generated from a tensor product $\mathbf{2} \otimes \mathbf{2}$ or $\mathbf{2} \otimes \mathbf{2}^*$ and can be expressed in a 2×2 matrix. For instance, consider the case $\mathbf{2} \otimes \mathbf{2}^* = \mathbf{3} \oplus \mathbf{1}$. Let be two doublets $q \sim (\mathbf{2}, Y_1)$ and $r \sim (\mathbf{2}, Y_2)$ that shape a triplet $(\mathcal{T}_1)_{ab}$ from the traceless hermitian part of $q_a r_b$. In matrix representation, the doublets transform as $q' = U_Y U_L q$ and the triplet,

$$\begin{aligned}
(\mathcal{T}_1)'_{ab} &= (U_Y (U_L)_{aa'} q_{a'})(U_Y (U_L)_{bb'} r_{b'})^\dagger \\
&\approx (1 + ie\frac{Y_1}{2} + \mathcal{O}(e^2))(1 - ie\frac{Y_2}{2} + \mathcal{O}(e^2))(\mathbb{1}_2 + ie\alpha T_{3L} + \mathcal{O}(e^2))(qr^\dagger)(\mathbb{1}_2 - ie\alpha T_{3L} + \mathcal{O}(e^2)) \\
&\approx (1 + ie\left(\frac{Y_1 - Y_2}{2}\right) + \mathcal{O}(e^2))(\mathcal{T}_1 + ie\alpha(T_{3L}\mathcal{T}_1 - \mathcal{T}_1 T_{3L}) + \mathcal{O}(e^2)) \\
&\approx \mathcal{T}_1 + ie\left([\alpha T_{3L}, \mathcal{T}_1] + \frac{Y}{2}\mathcal{T}_1\right) + \mathcal{O}(e^2),
\end{aligned} \tag{1.23}$$

where $Y = Y_1 - Y_2$ and hence its electric charge operator expanding up to the first order in e is

$$Q\mathcal{T}_1 = [\alpha T_{3L}, \mathcal{T}_1] + \frac{Y}{2}\mathcal{T}_1 = \begin{pmatrix} \frac{Y}{2} & \alpha + \frac{Y}{2} \\ -\alpha + \frac{Y}{2} & \frac{Y}{2} \end{pmatrix}. \tag{1.24}$$

It could be used for flavor symmetries where the symmetry also belongs to the group $SU(2)_V$. For example, if we choose the quark doublet $q = r = \begin{pmatrix} u & d \end{pmatrix}^T \sim (\mathbf{2}, 1/3)$ to build a composite meson, then $Y = Y(q) - Y(r) = 0$ and we have the real pion-like triplet $(\pi^+ \, \pi^0 \, \pi^-)^T$ (see (D.2)). To show that, let us go back to the product $\mathbf{2} \otimes \mathbf{2}^* = \mathbf{3} \oplus \mathbf{1}$. Using tensor notation, te doublets multiplication can be symmetrized by $q_i q^j = \left(q_i q^j - \frac{1}{2}\delta^i_j q_k q^k\right) + \frac{1}{2}\delta^i_j q_k q^k$. The upper index denotes an anti-doublet and the traceless part in parentheses represents a triplet.

Since $q = \begin{pmatrix} u & d \end{pmatrix}^T$ is the doublet then $\bar{q} = \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}^T$[3] is the anti-doublet. If we build the π-meson triplet $\mathcal{T}$ from the tensor defined above,

$$\begin{aligned}
\mathcal{T}_1^1 &= u\bar{u} - \frac{1}{2}\left(u\bar{u} + d\bar{d}\right) = \frac{1}{2}\left(u\bar{u} - d\bar{d}\right) = \frac{\pi^0}{\sqrt{2}}; \\
\mathcal{T}_1^2 &= u\bar{d} = \pi^+; \; T_2^1 = d\bar{u} = \pi^-; \\
\mathcal{T}_2^2 &= d\bar{d} - \frac{1}{2}\left(u\bar{u} + d\bar{d}\right) = -\frac{1}{2}\left(u\bar{u} - d\bar{d}\right) = -\frac{\pi^0}{\sqrt{2}}.
\end{aligned} \tag{1.25}$$

It obtains $\mathcal{T} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}$ as usual for adjoint $SU(2)$ triplets like the vector bosons in SM.

As we said before, for triplets in $SU(2) \otimes U(1)_Y$ there is another possibility, if $\mathbf{2} \otimes \mathbf{2} = \mathbf{3}_S \oplus \mathbf{1}$ where a second triplet $(\mathcal{T}_2)_{ab}$ is made up from the symmetric part of $q_a r_b$. Doing the same calculation as in the equation (1.23), we deduce the transformation $\mathcal{T}_2' \to U_Y U_L \mathcal{T}_2 U_L^T$ and

$$Q\mathcal{T}_2 \approx \{\alpha T_3, \mathcal{T}_2\} + \frac{Y}{2}\mathcal{T}_2 \approx \begin{pmatrix} \alpha + \frac{Y}{2} & \frac{Y}{2} \\ \frac{Y}{2} & -\alpha + \frac{Y}{2} \end{pmatrix}, \tag{1.26}$$

where $Y = Y_1 + Y_2$. Let us notice that with a rotation, the triplets in (1.24) and (1.26) are the same. There is no application for this type of triplet because of mesons = quark + antiquark (q and r transform differently) and does not exist composite particles quark + quark. Furthermore, leptons do not join together to form another particle, as far as we know. However it is interesting to notice that $\mathbf{2} \otimes \mathbf{2}$ produces different charge operator in (1.24) from $\mathbf{2} \otimes \mathbf{2}^*$ in (1.26).

Finally, let us deal with a quartet $\mathcal{F}$ which can be obtained from the product of three different doublets, $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4}$. $(\mathcal{F})_{abc}$ denotes the totally symmetric 3-rank tensor of

[3]We are considering $2 \otimes 2^* \neq 2 \otimes 2$ as a consequence of a tensor treatment and the anti-doublet is not arranged as usual in $SU(2)$.

tensor multiplication $q_a r_b s_c$ or

$$(\mathcal{F})_{abc} = \frac{1}{6}\left[q_a r_b s_c + q_b r_a s_c + q_b r_c s_a + q_c r_b s_a + q_c r_a s_b + q_a r_c s_b\right] \equiv \frac{1}{6} q_{(a} r_b s_{c)} \tag{1.27}$$

This tensor has eight components (octet). In order to make a four-element 2×2 matrix that fits into de SM Lagrangian coupling with other standard multiplets. Let us consider $q_a = r_a = s_a \sim (\mathbf{2}, Y)$ as a fermionic tensor, with which we can set $\mathcal{F}_{111} = q_1 q_1 q_1$, $\mathcal{F}_{112} = \mathcal{F}_{121} = \mathcal{F}_{211} = \frac{1}{3}(q_1 q_1 q_2 + q_1 q_2 q_1 + q_2 q_1 q_1)$, $\mathcal{F}_{122} = \mathcal{F}_{212} = \mathcal{F}_{221} = \frac{1}{3}(q_1 q_2 q_2 + q_2 q_1 q_2 + q_2 q_1 q_1)$ and $\mathcal{F}_{222} = q_2 q_2 q_2$ reducing the number of elements to four (quartet). Ordering the remaining components, we have $\mathcal{F} = \begin{pmatrix} \mathcal{F}_{111} & \mathcal{F}_{122} \\ \mathcal{F}_{112} & \mathcal{F}_{222} \end{pmatrix}$. Then, this quartet transforms as

$$\begin{aligned}
\mathcal{F}'_{abc} &= U_Y \left[(U_L)_a^\alpha q_\alpha (U_L)_b^\beta q_\beta (U_L)_c^\theta q_\theta\right] \\
&\approx \left(1 + i\frac{Y}{2}\right)\left[(\delta_a^\alpha + i\alpha(T_{3L})_a^\alpha)(\delta_b^\beta + i\alpha(T_{3L})_b^\beta)(\delta_c^\theta + i\alpha(T_{3L})_c^\theta)\right] q_\alpha q_\beta q_\theta \\
&\approx \left\{\delta_a^\alpha \delta_b^\beta \delta_c^\theta + i\left[\alpha\left(\delta_a^\alpha \delta_b^\beta (T_{3L})_c^\theta + \delta_b^\beta \delta_c^\theta (T_{3L})_a^\alpha + \delta_a^\alpha \delta_c^\theta (T_{3L})_b^\beta\right) + \frac{Y}{2}\delta_a^\alpha \delta_b^\beta \delta_c^\theta\right]\right\} q_\alpha q_\beta q_\theta \\
&\approx \mathcal{F}_{abc} + i\left[\alpha\left((T_{3L})_c^\theta \mathcal{F}_{ab\theta} + (T_{3L})_a^\alpha \mathcal{F}_{\alpha bc} + (T_{3L})_b^\beta \mathcal{F}_{a\beta c}\right) + \frac{Y}{2}\mathcal{F}_{abc}\right]
\end{aligned} \tag{1.28}$$

Then, the electric charge operator, taken into account the four mentioned components only, is

$$Q(\mathcal{F}_{abc}) = \alpha\left((T_{3L})_c^\theta \mathcal{F}_{ab\theta} + (T_{3L})_a^\alpha \mathcal{F}_{\alpha bc} + (T_{3L})_b^\beta \mathcal{F}_{a\beta c}\right) + \frac{Y}{2}\mathcal{F}_{abc} = \frac{1}{2}\begin{pmatrix} 3\alpha + Y & -\alpha + Y \\ \alpha + Y & -3\alpha + Y \end{pmatrix}. \tag{1.29}$$

For instance, let us take the case $\alpha = 1$ and $q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^T = \begin{pmatrix} u & d \end{pmatrix}^T \sim (\mathbf{2_L}, 1/3)$, then $Y = 1/3 + 1/3 + 1/3 = +1$. The Δ baryon quartet may be built from its charge operator,

$$Q(\mathcal{F}) = \begin{pmatrix} +2 & 0 \\ +1 & -1 \end{pmatrix}, \tag{1.30}$$

which corresponds to

$$\begin{aligned}
&\mathcal{F}_{111} = q_1 q_1 q_1 = uuu = \Delta^{++}, \quad \mathcal{F}_{112} = q_1 q_1 q_2 = uud = \Delta^{+}, \\
&\mathcal{F}_{122} = q_1 q_2 q_2 = udd = \Delta^{0}, \qquad \mathcal{F}_{222} = q_1 q_1 q_1 = ddd = \Delta^{-}
\end{aligned} \tag{1.31}$$

It can be noticed that the $U(1)$ charges (electric and hypercharge) are summatives.

In a flavor symmetry, this multiplet could be the Δ-baryon quadruplet of the quark model if $q_1 = u$ and $q_2 = d$.

Let us perform some other examples of exotic representations of $SU(2) \otimes U(1)_Y$ using the charge operator given in Eq. (1.22) and (1.24):

1. Model with $\alpha = 2$ and $Y = 0$. The electric charge eigenvalues for fields in the fundamental representation are $(+1, -1)$. For instance, a vector-like Dirac fermionic doublet $\begin{pmatrix} E^+ & E^- \end{pmatrix}^T$ or scalar $\begin{pmatrix} Y^+ & S^- \end{pmatrix}$, which could be coupled to a real vector triplet $\mathcal{T}$, with electric charge content

$$Q(\mathcal{T}_1) \approx [2T_3, \mathcal{T}_1] = \begin{pmatrix} 0 & +2 \\ -2 & 0 \end{pmatrix}, \quad Q(\mathcal{T}_2) \approx \{2T_3, \mathcal{T}_2\} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \tag{1.32}$$

   hence we could take $\mathcal{T}_1 = \begin{pmatrix} V^{++} \, V'^0 \, V^{--} \end{pmatrix}^T$. Notice that these fields may be introduced in extended gauge symmetries at higher energies.

2. Model with $\alpha = 2$ and $Y = 2$ obtains doublets with electric charge eigenvalues $(+2, 0)$ and triplets with $(+2, 0, -2)$ in the commutator case and $(+4, 2, 0)$ in the anti-commutator case. Here the triplets are different because the definition of doublets hypercharges for the commutator and anti-commutator case are different from each other.

3. The more exotic case when $\alpha = 1$ and $Y = -3$ include doublets with electric charge $(-1, -2)$. In this case it is possible to have doublets as $(\chi^- \, \chi^{--})^T$ which could be coupled with vector bosons like $(V^+ \, V^0 \, V^-)^T$ (SM). On the other hand, it is possible a complex scalar triplet like the one shown in Ref. [29] with $Y = -6$,

$$QH \approx [T_3, H] - 3H = \begin{pmatrix} -3 & +1 \\ -1 & -3 \end{pmatrix}. \tag{1.33}$$

All these exotic multiplets are possibilities within the SM symmetry group. However, there are other reasonable questions which can not be absolved and require to extend the mentioned gauge group. For example, why the three families? or why the representations are different for right-handed particles and left-handed ones?. Next chapters go beyond the SM symmetry group and discuss electric charge operators that belong to these extensions.

1.5 Electric charge and gauge couplings

Finally, electric charge might be viewed as coupling constant in $U(1)_Q$ symmetry after the breakdown. Then, there must have a relation with the other coupling constants in $SU(2)_L \otimes U(1)_Y$ symmetry in order to perform the SSB from the charge point of view.

In that way, to perform the mentioned relation, recall that coupling constants appear in the covariant derivative for doublets which is defined as

$$D^\mu \Phi = \partial^\mu \Phi + \left(i g_L W^{\mu j} \frac{\sigma_j}{2} + i g_Y \frac{Y}{2} B^\mu \mathbb{1}_2 \right) \Phi,$$

where $\frac{\sigma_j}{2}$, ($i = 1, 2, 3$) and $\frac{Y}{2}$ are the generators of $SU(2)_L$ and $U(1)_Y$ groups respectively.

By developing the kinetic part of the Higgs sector in the SSB context, $[D^\mu \Phi]^\dagger [D_\mu \Phi]$, vector boson masses and relations between gauge coupling constant with weak mixing angle are obtained.

$$M_W = \frac{g_L v}{2}, \; M_Z = \frac{v}{2}\sqrt{g_L^2 + g_Y^2}, \; M_A = 0 \tag{1.34}$$

$$\frac{M_W^2}{M_Z^2} = \frac{g_L^2}{g_L^2 + g_Y^2} = \cos^2 \theta_W. \tag{1.35}$$

But also, there is a way to find the relations with electric charge from comparing the coefficients of the interaction of lepton charged current with photons in SM symmetry and QED symmetry, obtaining $e = g_Y \cos\theta_W = g_L \sin\theta_W$ (see appendix (E)). Then

$$\frac{1}{e^2} = \frac{1}{g_L^2} + \frac{1}{g_Y^2}, \tag{1.36}$$

where g_L stands for $SU(2)_L$ and g_Y for $U(1)_Y$. This relation was obtained at low energies, since the coefficient comparison took place after SSM, below the electroweak scale.

For $\sin^2 \theta_W \approx 0.2315$ [1] and $e^2 = 4\pi\alpha = \frac{4\pi}{137}$, then

$$g_L \approx 0.6295, \quad g_Y \approx 0.3455. \tag{1.37}$$

Chapter 2

Electroweak model $\mathrm{SU(3)_L \otimes U(1)_N}$

From what was stated previously about of the lack of robustness of the SM to explain some facts such as neutrino mass, hierarchy problem, matter/antimatter asymmetry, etc. (see the introduction), physicists have been conjecturing for decades, new gauge extended symmetries to that of the SM which reveal new physics. These models predict the existence of new exotic fermions as well as new gauge bosons and scalars that depend on the symmetry group they are defined. Some of these exotic particles are being searched at the LHC [30]. In some extensions, it is possible to understand the number of families from the cancellation of anomalies [17]. This is the case of the 331 models in $SU(3)_C \otimes SU(3)_L \otimes U(1)_N$ gauge symmetry which includes to SM whose particle content, in addition to new ones, is reproduced after the SSB.

As in the equation (1.3), the charge relation [31] can be defined as

$$\frac{Q}{e} = \alpha T_{3L} + \beta T_{8L} + N, \tag{2.1}$$

where T_{3L} and T_{8L} are diagonal generators of the $SU(3)_L$ group. The hypercharge N, as Y in the SM, unifies weak and electromagnetic interactions above the electroweak energy scale.

It should be pointed out that the value of β defines an entire theory in 331 models and the β that reproduces the same lepton content as in the SM, is the one where no exotic lepton is considered in leptonic triplets. It occurs in the called minimal 331 model with $\beta = -\sqrt{3}$ when only neutrinos and known charged leptons are considered.

2.1 Lagrangian

The electroweak Lagrangian in this symmetry is similar to that of the SM in eq. (1.1)

$$\begin{aligned}
\mathscr{L} &= -\tfrac{1}{4}F_{i\mu\nu}F_i^{\mu\nu} - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} && \ldots \text{Vector sector} \\
&+i\overline{\Psi}_{iL}\not{D}\Psi_{iL} + i\overline{\Psi}_{iR}\not{D}\Psi_{iR} + h.c. && \ldots \text{Fermion-Vector sector} \\
&+\overline{\Psi}_{iL}y^a_{ij}\Psi_{jR}\Phi^a + \overline{\Psi}_{iL}w_{ij}(\Psi_{jL})^c S + h.c. && \ldots \text{Interaction sector (Yukawa)} \\
&+ (D_\mu\Phi)^\dagger (D^\mu\Phi) - V(\Phi) && \ldots \text{Scalar-Vector sector}
\end{aligned} \tag{2.2}$$

There are some aspects of this Lagrangian we must point out:

- In the vector sector there are eight tensors $F_{i\mu\nu} = \partial_\mu W_{i\nu} - \partial_\nu W_{i\mu} + g_{3L} f_{ijk} W_{j\mu} W_{k\nu}$, $i = 1\ldots 8$ whose vector fields $W_{i\mu}$ are associated to $SU(3)_L$ generators (f_{ijk} are the structure constant of this Lie group) and one tensor $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ of which B_μ is associated to $U(1)_N$ generator.

- In the fermion sector triplets (or anti-triplets defined for the second and third family of quarks) Ψ_{iL}, $i = 1\ldots 3$ are representations of the left-handed leptons or quarks, while Ψ_{iR} are the corresponding right-handed singlets.

- In the Yukawa sector there are three different scalar triplets Φ^a, $a = 1\ldots 3$, and one sextet S. All these multiplets (or anti-triplets) couple with fermions. These couplings are written to take into account after breakdown two classes of mass terms, the first adding decants only into Dirac mass terms $m\overline{\varphi_L}\varphi_R + h.c.$ for all the fermions, while the second adding into Majorana mass terms $m\overline{\varphi_L}(\varphi_L)^c + h.c.$ for neutrinos [32], although it also obtains Dirac mass terms for charged leptons in some cases. Then, the following must be fulfilled for the scalar multiplets Φ:

$$\overline{\Psi_{iL}}\Psi_{jR}\Phi : \begin{cases} 3^* \otimes 1 \otimes \Phi = 1 \Rightarrow \Phi = 3 \\ 3 \otimes 1 \otimes \Phi = 1 \Rightarrow \Phi = 3^* \end{cases}$$

$$\overline{\Psi_{iL}}(\Psi_{jL})^c\Phi : \begin{cases} 3^* \otimes 3^* \otimes \Phi = 1 \Rightarrow \Phi = 3 \otimes 3 = 3^* \oplus 6 \\ 3 \otimes 3 \otimes \Phi = 1 \Rightarrow \Phi = 3^* \otimes 3^* = 3 \oplus 6^* \end{cases}$$

which means that the usual Dirac mass terms are obtained if Φ is a triplet (or an anti-triplet) while the Majorana mass terms are achieved with Φ being an anti-triplet (triplet) or a sextet (anti-sextet).

- From the last point, scalar multiplets could be triplets or sextets as possible representations. If it is a sextet S, its corresponding kinetic term is written in the shape of a Hilbert-Schmidt inner product, $\mathrm{Tr}\left[(D^\mu S)^\dagger (D_\mu S)\right]$. The potential $V(\Phi)$ is defined as generally as possible according to the symmetries (including the discrete ones if aplicable) added to the theory [33].

In the Lagrangian density described above, covariant derivatives for fermion triplets Ψ_{iL} and singlets Ψ_{iR} are

$$\begin{aligned} D^\mu \Psi_{iL} &= \left[\partial^\mu + i g_{3L} T_j W_j^\mu + i g_N B^\mu N_L\right] \Psi_{iL}, \\ D^\mu \Psi_{iR} &= \left[\partial^\mu + i g_N B^\mu N_R\right] \Psi_{iR}, \end{aligned} \tag{2.3}$$

where:

- Ψ_{iL} is the left-handed fermion triplet: $\Psi_{iL} = L_i$ for leptons and $\Psi_{iL} = \mathcal{Q}_{iL}$ for quarks (for anti-quarks $i = 1, 2$ the derivative changes to its complex conjugated form). Ψ_{iR} is the right-handed fermion singlet and could be the lepton singlet R_i or the quark singlet $\mathcal{Q}_{iR}$. Index i stands for the three families $i = 1, 2, 3$.

- g_{3L} y g_N are the coupling constants of $SU(3)_L$ and $U(1)_N$ respectively.

- N_L y N_R are hypercharges of left- and right-handed fields.

- W_j^μ with $j = 1, ..., 8$ are the eight gauge fields that correspond to $SU(3)_L$ generators and B^μ is the gauge field of $U(1)_N$ group.

Like the SM with its $SU(2)_L$ gauge bosons, the eight gauge bosons of $SU(3)_L$ transform in the adjoint representation and mix themselves to form the mass (physical) states (see appendix (B).

Left-handed fermions triplets and right-handed singlets transform as

$$\begin{aligned} &\Psi_L \to \Psi'_{iL} = U_N U_L \Psi_{iL}, \\ &\Psi_R \to \Psi'_{iR} = U_N \Psi_{iR}, \end{aligned} \tag{2.4}$$

where $U_L = \exp\left[i g_{3L} T_j \omega_j(x)\right]$ and $U_N = \exp\left[i g_N N f(x)\right]$. All these triplets and singlets are grouped in the same three families of SM with an additional degree of freedom for each family.

The gauge fields transform as

$$
\begin{aligned}
W_i^\mu(x) &\rightarrow W_i^{\mu'}(x) = W_i^\mu(x) - \partial^\mu \omega_i(x) - g_{3L} f_{ijk}\omega_j(x) W_k^\mu(x), \\
B^\mu(x) &\rightarrow B^{\mu'}(x) = B^\mu(x) - \partial^\mu f(x),
\end{aligned} \tag{2.5}
$$

where $f(x)$ y $\omega_j(x)$ are real function-like parameters, numbers $f_{ijk} = \frac{1}{4i}\,\mathrm{Tr}\{[\lambda_i, \lambda_j]\,\lambda_k\}$, $i, j, k = 1 \ldots 8$ are structure constants defined for $SU(3)_L$ group and λ_i are the Gell-Mann matrices.

The covariant derivatives for scalar multiplets are

$$
\begin{aligned}
D^\mu \Phi &= \left[\partial^\mu + i g_{3L} T_j W_j^\mu + i g_N B^\mu N_\Phi\right] \Phi, \\
D^\mu S &= \partial^\mu S + i g_{3L}\left[T_j W_j^\mu S + S (T_j W_j^\mu)^T\right].
\end{aligned} \tag{2.6}
$$

The gauge covariant derivative for the sextet has a peculiar form without a commutator, which is usual for matrix representations, because of the way it is constructed (See (F.13) in Appendix (F)). This scalar multiplet is formed by two triplets as may be seen in (2.25). In short, $S = \eta\eta^T$ and this relation yields the way it transforms.

The gauge transformations are

$$
\begin{aligned}
&\Phi \rightarrow \Phi' = U_N U_L \Phi, \\
&S \rightarrow S' = U_N U_L S U_L^T.
\end{aligned} \tag{2.7}
$$

These multiplet transformations, as well as their covariant derivatives, make the Lagrangian remain gauge invariant. In the sextet case, some possible allowed terms may be $G^S_{ab}\overline{f^a_L}(f^b_L)^c S$ [34] to give neutrino mass via coupling with lepton triplets f^a_L or interactions with other scalars inside the potential.

In the next section, the particle content is described separately in the fermionic, bosonic and scalar sector of the lagrangian; all of them in terms of their electric charge operator.

2.2 Electric charge for Fermions

Here, the main difference with EWSM is that left fermions are $SU(3)_L$ triplets which possess, in addition to the known leptons, a third new component (one for each family) and its nature depends on the values of α and β in eq. (2.1). Models with $\alpha = 1$ and $\beta = -\sqrt{3}$ are considered in Refs. [32, 35, 36] while those with $\alpha = 1$ and $\beta = -(1/\sqrt{3})$ in Refs. [37–39].

In general, triplets $\mathcal{T}$ obtain their electric charge eigenvalues directly from equation (2.1) as the fundamental representation of the symmetry group. In this case, SSB is easily visualized when the gauge transformation itself goes from $U_N U_L$ to U_Q in the breaking process.

$$\begin{aligned} U_L U_N \mathcal{T} &= e^{iT_j \omega_j(x)} e^{iNf(x)} \mathcal{T} \\ &\approx \left[\mathbf{1}_3 + ie\left(\alpha T_{3L} + \beta T_{8L}\right) + ieN\mathbf{1}_3\right] \mathcal{T} \\ &\approx \mathcal{T} + ieQ\mathcal{T}, \text{ and} \\ U_L^\dagger U_N^* \mathcal{T}^* &\approx \left[\mathbf{1}_3 - ie\left(\alpha T_{3L} + \beta T_{8L}\right) - ieN\mathbf{1}_3\right] \mathcal{T}^* \\ &\approx \mathcal{T}^* + ieQ\mathcal{T}^*, \end{aligned} \tag{2.8}$$

where in the second line $\omega_j(x) = 0$ for $j = 1, 2, 4, \ldots 7$ and $\omega_j(x) = f(x) = e$ for $i = 1, 3$. This is obtained thanks to the gauge invariance of the vacuum expectation values of the theory, just like in the SM (see appendix (A)). The resulting electric charge operator is

$$\begin{aligned} Q(\mathcal{T}) &= \frac{1}{2}\mathrm{diag}(\alpha + \frac{\beta}{\sqrt{3}} + 2N, -\alpha + \frac{\beta}{\sqrt{3}} + 2N, -2\frac{\beta}{\sqrt{3}} + 2N)\mathcal{T} \\ Q(\mathcal{T}^*) &= \frac{1}{2}\mathrm{diag}(-\alpha - \frac{\beta}{\sqrt{3}} + 2N', \alpha - \frac{\beta}{\sqrt{3}} + 2N', 2\frac{\beta}{\sqrt{3}} + 2N')\mathcal{T}^* \end{aligned} \tag{2.9}$$

The symmetry group breaks down as follows: $SU(3)_L \otimes U(1)_N \xrightarrow{\langle\chi\rangle} SU(2)_L \otimes U(1)_Y \xrightarrow{\langle\eta\rangle,\langle\rho\rangle} U(1)_Q$. In the first breakdown, the fermion triplets decant in the SM fermion doublets. To do so, an ad-hoc scalar triplet χ is coupled with lepton and quark currents in the Yukawa sector [39]. For models with exotic lepton in leptonic triplet, this scalar gives mass only to the exotic fermion X_i or J_i since the vacuum expectation value (vev) is located in its third component. Then, fermion components are

$$L_i = \begin{pmatrix} \nu_{iL} \\ e_{iL} \\ X_{iL} \end{pmatrix} \xrightarrow{\langle\chi\rangle} \begin{pmatrix} \nu_{iL} \\ e_{iL} \end{pmatrix}, \ \mathcal{Q}_{1L} = \begin{pmatrix} u_{1L} \\ d_{1L} \\ J_{1L} \end{pmatrix} \xrightarrow{\langle\chi\rangle} \begin{pmatrix} u_{1L} \\ d_{1L} \end{pmatrix}, \mathcal{Q}_{jL} = \begin{pmatrix} -d_{jL} \\ u_{jL} \\ J_{jL} \end{pmatrix} \xrightarrow{\langle\chi\rangle} \begin{pmatrix} -d_{jL} \\ u_{jL} \end{pmatrix} \tag{2.10}$$

with $i = 1, 2, 3$ and $j = 2, 3$. Let us notice that $\mathcal{Q}_{jL}$ goes to an $SU(2)$ anti-doublet which transforms in the same way as a doublet. Besides, in this sort of models, the lepton triplet set up the component X_{iL} as an exotic lepton or a charge conjugated field in electron family, both of them have the same electric charge.

Moreover, before the breakdown, quark triplets have two representations: $\mathcal{Q}_{1L}$ and $\mathcal{Q}_{jL}$ which transform like $\mathcal{T}$ and $\mathcal{T}^*$ respectively.

In order to determine hypercharges, values of α must be deduced. The electric charges for known SM leptons and quarks in (2.10) are achieved only if $\alpha = 1$ in (2.9):

$$
\begin{aligned}
L_i^{SM} = \begin{pmatrix} \nu_{iL} \\ e_{iL} \end{pmatrix} : \quad & \left.\begin{aligned} q(\nu_{iL}) &= \frac{1}{2}\alpha + \frac{\beta}{2\sqrt{3}} + N(L_i) = 0 \\ q(e_{iL}) &= -\frac{1}{2}\alpha + \frac{\beta}{2\sqrt{3}} + N(L_i) = -1 \end{aligned}\right\} \quad \alpha = 1 \\
Q_{1L}^{SM} = \begin{pmatrix} u_{1L} \\ d_{1L} \end{pmatrix} : \quad & \left.\begin{aligned} q(u_{1L}) &= \frac{1}{2}\alpha + \frac{\beta}{2\sqrt{3}} + N(Q_{1L}) = \frac{2}{3} \\ q(d_{1L}) &= -\frac{1}{2}\alpha + \frac{\beta}{2\sqrt{3}} + N(Q_{1L}) = -\frac{1}{3} \end{aligned}\right\} \quad \alpha = 1 \\
Q_{jL}^{SM} = \begin{pmatrix} -d_{jL} \\ u_{jL} \end{pmatrix} : \quad & \left.\begin{aligned} q(d_{jL}) &= -\frac{1}{2}\alpha - \frac{\beta}{2\sqrt{3}} + N(Q_{jL}) = -\frac{1}{3} \\ q(u_{jL}) &= \frac{1}{2}\alpha - \frac{\beta}{2\sqrt{3}} + N(Q_{jL}) = \frac{2}{3} \end{aligned}\right\} \quad \alpha = 1
\end{aligned}
$$

For this α-value the corresponding hypercharges are

$$N(L_i) = -\frac{1}{2} - \frac{\beta}{2\sqrt{3}}, \quad N(\mathcal{Q}_{1L}) = \frac{1}{6} - \frac{\beta}{2\sqrt{3}} \quad \text{and} \quad N(\mathcal{Q}_{jL}) = \frac{1}{6} + \frac{\beta}{2\sqrt{3}}. \tag{2.11}$$

Then,

$$
\begin{aligned}
&Q(L_i) = \text{diag}\left(0, -1, -\frac{1}{2} - \frac{\sqrt{3}\beta}{2}\right), \quad Q(\mathcal{Q}_{1L}) = \text{diag}\left(+\frac{2}{3}, -\frac{1}{3}, \frac{1}{6} - \frac{\sqrt{3}\beta}{2}\right), \\
&Q(\mathcal{Q}_{jL}) = \text{diag}\left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{6} + \frac{\sqrt{3}\beta}{2}\right).
\end{aligned} \tag{2.12}
$$

If we state that the electric charge of the third component of lepton triplets is $q \equiv q(X_i) = q(e_i^c) = -\frac{1}{2} - \frac{\sqrt{3}\beta}{2}$, then $q(J_{1L}) = q + \frac{2}{3}$ and $q(J_{jL}) = -q - \frac{1}{3}$. The fermion multiplets are described in the following table,

	L_i	$\mathcal{Q}_{1L}$	$\mathcal{Q}_{jL}$
$SU(3)_C$	1	3	3
$SU(3)_L$	3	3	3^*
N	$\frac{q-1}{3}$	$\frac{q+1}{3}$	$-\frac{q}{3}$
Q	$\begin{pmatrix} 0 \\ -1 \\ q \end{pmatrix}$	$\begin{pmatrix} +2/3 \\ -1/3 \\ q+2/3 \end{pmatrix}$	$\begin{pmatrix} -1/3 \\ +2/3 \\ -q-1/3 \end{pmatrix}$

Table 2.1: Representation of fermion triplets. The first two rows show the dimension in $SU(3)_C$ and $SU(3)_L$ symmetries.

The right-handed fermions transform as singlets and have the same electric charge of their left-handed partners when the symmetry has lowered to $U(1)_Q$.

From the table (2.2), triplets are interdependent in terms of their electric charges and hypercharges. It means that the whole theory depends upon the possible $\beta = -(\frac{2q+1}{\sqrt{3}})$ values. In short,

β	$\sqrt{3}$	$1/\sqrt{3}$	$-\sqrt{3}$	$-1/\sqrt{3}$
L	$\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \sim -1$	$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \sim -2/3$	$\begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix} \sim 0$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \sim -1/3$
$\mathcal{Q}_{1L}$	$\begin{pmatrix} +2/3 \\ -1/3 \\ -4/3 \end{pmatrix} \sim -1/3$	$\begin{pmatrix} +2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \sim 0$	$\begin{pmatrix} +2/3 \\ -1/3 \\ +5/3 \end{pmatrix} \sim +2/3$	$\begin{pmatrix} +2/3 \\ -1/3 \\ +2/3 \end{pmatrix} \sim +1/3$
$\mathcal{Q}_{jL}$	$\begin{pmatrix} -1/3 \\ +2/3 \\ +5/3 \end{pmatrix} \sim +2/3$	$\begin{pmatrix} -1/3 \\ +2/3 \\ +2/3 \end{pmatrix} \sim +1/3$	$\begin{pmatrix} -1/3 \\ +2/3 \\ -4/3 \end{pmatrix} \sim -1/3$	$\begin{pmatrix} -1/3 \\ +2/3 \\ -1/3 \end{pmatrix} \sim 0$

Table 2.2: Electric charge eigenvalues for fermionic triplets with their respective hypercharges on the right side.

It may be noted that theories with no exotic electric charges in their third component triplets have $\beta = \pm\frac{1}{\sqrt{3}}$, however models with $\beta = -\sqrt{3}$ may consider a lepton triplet with a charge conjugated lepton in its third component which means that there is no exotic charge. Nevertheless, in all models could be electric charges of unknown very massive particles that are yet to be discovered.

After the symmetry breakdown which leads toward the $U(1)_Q$ symmetry, the exotic fermions $X_{iL,R}$ and $J_{i,L,R}$, $i = 1, 2, 3$ are accommodated in mass terms within the respective Lagrangian. Here, these fields have their respective electric charge (via Noether's theorem) despite having predicted them before the breaking. As the "future" electric charges are wanted to be the same in both theories, 331 and SM, then we can equate the relations (1.3) and (2.1) to obtain a new relation for hypercharges. With $\alpha = 1$,

$$\begin{aligned} Y\mathbf{1}_3 &= 2(\beta T_{8L} + N\mathbf{1}_3) = \text{diag}\left(\frac{\beta}{\sqrt{3}} + 2N, \frac{\beta}{\sqrt{3}} + 2N, -\frac{2\beta}{\sqrt{3}} + 2N\right), \\ Y^*\mathbf{1}_3 &= 2(-\beta T_{8L} + N^*\mathbf{1}_3) = \text{diag}\left(-\frac{\beta}{\sqrt{3}} + 2N^*, -\frac{\beta}{\sqrt{3}} + 2N^*, \frac{2\beta}{\sqrt{3}} + 2N^*\right), \end{aligned} \tag{2.13}$$

where N^* and Y^* stands for anti-triplet hypercharges.

With the N-values given in (2.11), the SM doublets are recovered with their respective Y-values. Also, the hypercharges for exotic fermions can be obtained $Y(X_{iL}) = -1 - \sqrt{3}\beta = 2q$, $Y(J_{1L}) = \frac{1}{3} - \sqrt{3}\beta = 2q + \frac{4}{3}$ and $Y(J_{jL}) = \frac{1}{3} + \sqrt{3}\beta = -2q - \frac{2}{3}$. Here, X_{i_L} are the exotic leptons and J_{i_L} are the exotic quarks that came from the third component/state of fermion triplets.

2.3 Electric charge for vector bosons

As in the SM, gauge bosons (nine in this case) are written in the $SU(3)$ adjoint representation as 3×3 matrices, see Appendix (B). Using $\alpha = 1$ and the same procedure as (1.15), the transformation of $\widetilde{W}_\mu = \lambda_j W_{j\mu}$ is

$$\widetilde{W}_\mu \to \widetilde{W}'_\mu \approx \widetilde{W}_\mu + ie\Big[T_{3L} + \beta T_{8L}, \widetilde{W}_\mu\Big] \approx \widetilde{W}_\mu + ieQ\widetilde{W}_\mu. \tag{2.14}$$

Then,

$$Q(\widetilde{W}_\mu) \approx \Big[T_{3L} + \beta T_{8L}, \widetilde{W}_\mu\Big] = \begin{pmatrix} 0 & +1 & \frac{1+\sqrt{3}\beta}{2} \\ -1 & 0 & \frac{-1+\sqrt{3}\beta}{2} \\ \frac{-1-\sqrt{3}\beta}{2} & \frac{1-\sqrt{3}\beta}{2} & 0 \end{pmatrix}_{\widetilde{W}} = \begin{pmatrix} 0 & +1 & -q \\ -1 & 0 & -(1+q) \\ q & 1+q & 0 \end{pmatrix}_{\widetilde{W}} . \tag{2.15}$$

It can be seen that electric charges of the SM bosons ($W^\pm$, Z^0), which are mixtures of $SU(2)_L \otimes U(1)_Y$ gauge symmetry bosons, can be obtained from the first 2×2 entries of (2.15) and are independent of β and/or q values. As said before, the choice of a particular β is necessary to construct the whole theory and each value of it gives a different particle content with new exotic bosons and the new physics that it entails. All possible β values obtain at least one fourth extra neutral boson (Z^0') and there could be more if $\beta = \pm\frac{1}{\sqrt{3}}$. There is no fundamental reason to prevent accepting any value defined for beta, the choice made for any of these values depends on the constrains one impose to develop the theory. For instance, if no exotic charge is required then we should choose $\beta = \pm\frac{1}{\sqrt{3}}$ or if only three neutral bosons are wanted, the ad-hoc choice is $\beta = \pm\sqrt{3}$.

	$\beta = \sqrt{3},\ q = -2$	$\beta = 1/\sqrt{3},\ q = -1$	$\beta = -\sqrt{3},\ q = +1$	$\beta = -1/\sqrt{3},\ q = 0$
$\widetilde{W}_\mu$	$\begin{pmatrix} 0 & +1 & +2 \\ -1 & 0 & +1 \\ -2 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & +1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & -2 \\ +1 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$

Table 2.3: Electric charge eigenvalues for gauge bosons

One of the most used 331 model proposes $\beta = -\sqrt{3}$ [35]. For this value we can define

$$\widetilde{W}_\mu = \begin{pmatrix} W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} & W_{1\mu} - iW_{2\mu} & W_{4\mu} - iW_{5\mu} \\ W_{1\mu} + iW_{2\mu} & -W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} & W_{6\mu} - iW_{7\mu} \\ W_{4\mu} + iW_{5\mu} & W_{6\mu} + iW_{7\mu} & -2\frac{W_{8\mu}}{\sqrt{3}} \end{pmatrix} \equiv \begin{pmatrix} W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} & \sqrt{2}W_\mu^+ & \sqrt{2}V_\mu^- \\ W_\mu^- & -W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}} & \sqrt{2}U_\mu^{--} \\ \sqrt{2}V_\mu^+ & \sqrt{2}U_\mu^{++} & -2\frac{W_{8\mu}}{\sqrt{3}} \end{pmatrix},$$

where the charged vector bosons are

$$\begin{aligned} W_\mu^\pm &= \frac{(W_{1\mu} \mp iW_{2\mu})}{\sqrt{2}}, \\ V_\mu^\pm &= \frac{(W_{4\mu} \pm iW_{5\mu})}{\sqrt{2}}, \\ U_\mu^{\pm\pm} &= \frac{(W_{6\mu} \pm iW_{7\mu})}{\sqrt{2}}. \end{aligned} \tag{2.16}$$

Their electric charges were already known from the table (2.3) for $\beta = -\sqrt{3}$. It is not necessary to know them from their interactions within the Lagrangian, which is what this work is about.

The neutral vector boson are (see in the Appendix eq. (E.24))

$$\begin{aligned} A_\mu &= \frac{1}{\sqrt{1+4t^2}}\left[\left(W_{3\mu} - \sqrt{3}\,W_{8\mu}\right)t + B_\mu\right] \\ Z_\mu &\approx \frac{1}{\sqrt{1+4t^2}}\left(\sqrt{1+3t^2}\,W_{3\mu} + \frac{\sqrt{3}t^2}{\sqrt{1+3t^2}}\,W_{8\mu} - \frac{t}{\sqrt{1+3t^2}}\,B_\mu\right) \\ Z'_\mu &\approx \frac{1}{\sqrt{1+3t^2}}\left(W_{8\mu} + \sqrt{3}t\,B_\mu\right), \end{aligned} \tag{2.17}$$

where $t \equiv \tan\theta = \dfrac{g_N}{g_{3L}} = \dfrac{\sin\theta_W}{\sqrt{1-4\sin^2\theta_W}}$ (E.22).

Solving for gauge bosons:

$$\begin{aligned} W_{3\mu} &\approx A_\mu \sin\theta_W + Z_\mu \cos\theta_W \\ W_{8\mu} &\approx -A_\mu \sin\theta_W \sqrt{3 - 12\sin^2\theta_W} + Z_\mu\sqrt{3}\sin\theta_W \tan\theta_W + Z'_\mu \sec\theta_W \sqrt{1-4\sin^2\theta_W} \\ B_\mu &\approx A_\mu\sqrt{1-4\sin^2\theta_W} - Z_\mu \tan\theta_W \sqrt{1-4\sin^2\theta_W} + Z'_\mu\sqrt{3}\tan\theta_W \end{aligned} \tag{2.18}$$

These results show that after breaking, $W_{3\mu}$ has the same transformation as the equivalent $W_{3\mu}$ of SM, but B_μ do not. It could be explained with the assumption that there are another intermediate degree of freedom Z'_μ which is presumed to be achieved only at very high energies where the Weinberg angle has run to a different value.

In that sense, if Z^0_μ (or $W^\pm_\mu$) mass values theoretically obtained in SM and 331 (see appendix E) symmetries are compared,

$$\begin{aligned} M_Z^2 &\overset{331}{=} \frac{g_{3L}^2}{2}\left(\frac{1+4t^2}{1+3t^2}\right)\left(v_\eta^2+v_\rho^2+v_2^2\right) \overset{\text{SM}}{=} \frac{v^2}{4}\left(g_L^2+g_Y^2\right) \\ M_W^2 &\overset{331}{=} \frac{g_{3L}^2}{2}\left(v_\eta^2+v_\rho^2+v_2^2\right) \overset{\text{SM}}{=} \frac{g_L^2 v^2}{4}, \end{aligned} \tag{2.19}$$

we can deduce a possible relation between expectation values and constant couplings,

$$v = \sqrt{2\left(v_\eta^2+v_\rho^2+v_2^2\right)}, \quad g_{3L} = g_L \quad \text{and} \quad g_N = \frac{g_L g_Y}{\sqrt{g_L^2 - 3g_Y^2}},$$

where, v and g stand for SM and v_η, v_ρ, v_2, g_{3L} and g_N for 331 symmetries.

Therefore,

$$\frac{1}{g_Y^2} = \frac{3}{g_L^2} + \frac{1}{g_N^2}. \tag{2.20}$$

As in the last chapter, the relations between symmetry constants and electric charge can be obtained from comparing the coefficients of the interaction of lepton charged current with photons in 331 symmetry and QED symmetry, (Appendix E)

$$e = g_{3L}\left(\frac{t}{\sqrt{1+4t^2}}\right) = \frac{g_{3L}\sin\theta}{\sqrt{1+3\sin^2\theta}} = \frac{g_N\cos\theta}{\sqrt{1+3\sin^2\theta}}. \tag{2.21}$$

Then,

$$e = \frac{g_N g_{3L}}{\sqrt{g_{3L}^2 + 4g_N^2}} \rightarrow \boxed{\frac{1}{e^2} = \frac{4}{g_{3L}^2} + \frac{1}{g_N^2}}. \tag{2.22}$$

For $\sin^2\theta_W \approx 0.2315$ [1] and $e^2 = 4\pi\alpha = \frac{4\pi}{137}$, then $t^2 = \frac{463}{148}$ and

$$g_{3L} = g_L \approx 0.6295, \quad g_N \approx 1.1133 \neq g_Y \approx 0.3455. \tag{2.23}$$

2.4 Electric charge for scalar bosons

The original model [35] features three scalar triplets. Each of them is built in such a way that there is one neutral scalar field inside at least. These scalars ρ, η and χ have electric charges with $\alpha = 1$ in equation (2.9),

	$\beta = \sqrt{3},\ q = -2$	$\beta = 1/\sqrt{3},\ q = -1$	$\beta = -\sqrt{3},\ q = +1$	$\beta = -1/\sqrt{3},\ q = 0$
η	$\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \sim -1$	$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \sim -\frac{2}{3}$	$\begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix} \sim 0$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \sim -\frac{1}{3}$
ρ	$\begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} \sim 0$	$\begin{pmatrix} +1 \\ 0 \\ 0 \end{pmatrix} \sim \frac{1}{3}$	$\begin{pmatrix} +1 \\ 0 \\ +2 \end{pmatrix} \sim 1$	$\begin{pmatrix} +1 \\ 0 \\ +1 \end{pmatrix} \sim +\frac{2}{3}$
χ	$\begin{pmatrix} +2 \\ +1 \\ 0 \end{pmatrix} \sim 1$	$\begin{pmatrix} +1 \\ 0 \\ 0 \end{pmatrix} \sim \frac{1}{3}$	$\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \sim -1$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \sim -\frac{1}{3}$

Table 2.4: Electric charge eigenvalues for scalar triplets with different values of β. The right numbers are N-hypercharges.

As seen in the last table, locations of one neutral scalar (Higgs) bosons in η, ρ, and χ are the same for each β. Furthermore, for $\beta = \pm\frac{1}{\sqrt{3}}$ only two triplets are necessary to be used in the theory [40] due to the same electric charge of ρ and χ, also a similar order of magnitud of their vev's (v_η and v_ρ). Besides, we have to be careful with $\beta = \pm\frac{1}{\sqrt{3}}$ due to the possibility of two vev's for some of their triplets. In those cases a new analysis must be promoted to achieve the vev relation between the SM and 331 theories as was done in the previous section. The whole scalar and interaction sector of the Lagrangian have a direct correspondence with each β-values.

On the other hand, a theory that includes Dirac and/or Majorana mass terms for neutrinos in order to implement the See-saw mechanism, needs an additional scalar sextet [32]. For any value of β (or q), the mentioned sextet is defined as

$$S = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_4 & \sigma_5 \\ \sigma_3 & \sigma_5 & \sigma_6 \end{pmatrix} \sim (\mathbf{6}_L, N). \tag{2.24}$$

In $SU(3)_L$, when two triplets (or antitriplets) are coupled, it is verified that $3 \otimes 3 = 3^*_A \oplus 6_S$ (or $3^* \otimes 3^* = 3_A \oplus 6^*_S$) [41]. In tensor form, be u^a, v^b, $a, b = 1, 2, 3$ two $SU(3)$ first-order tensors ($SU(3)$ triplets), then their multiplication may be represented in matrices form,

$$
\begin{aligned}
\Lambda^{ab} = u^a v^b &= \begin{pmatrix} u^1v^1 & u^1v^2 & u^1v^3 \\ u^2v^1 & u^2v^2 & u^2v^3 \\ u^3v^1 & u^3v^2 & u^3v^3 \end{pmatrix} \\
= \frac{1}{2} &\begin{pmatrix} 2u^1v^1 & u^1v^2 + u^2v^1 & u^1v^3 + u^3v^1 \\ u^2v^1 + u^1v^2 & 2u^2v^2 & u^2v^3 + u^3v^2 \\ u^3v^1 + u^1v^3 & u^3v^2 + u^2v^3 & 2u^3v^3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & u^1v^2 - u^2v^1 & u^1v^3 - u^3v^1 \\ u^2v^1 - u^1v^2 & 0 & u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 & u^3v^2 - u^2v^3 & 0 \end{pmatrix} \\
= \frac{1}{2} &\left(u^a v^b + u^b v^a\right) + \frac{1}{2}\epsilon^{abc}\epsilon_{cde} u^d v^e = \frac{1}{2}\left(\Lambda^{ab} + \Lambda^{ba}\right) + \frac{1}{2}\epsilon^{abc}\epsilon_{cde}\Lambda^{de}.
\end{aligned}
$$

It is evident that the first addend is 6_S and the second is 3^*_A. Then, the sextet may be expressed as $S^{ab} = \frac{1}{2}\left(\Lambda^{ab} + \Lambda^{ba}\right)$. Since triplets transform as $u^a \to u'^a = U^{ab} u^b$ then tensor Λ^{ab} is transformed as

$$
\begin{aligned}
\Lambda'^{ab} &= \left(U^{ac}u^c\right)\left(U^{bd}v^d\right) = \left(U^{ac}u^c\right)\left(v^d U^{db\,T}\right) \\
&\approx \left[\delta^{ac} + ie\left(T^{ac}_{3L} + \beta T^{ac}_{8L} + N_1\delta^{ac}\right)\right]\left(u^c v^d\right)\left[\delta^{db} + ie\left(T^{db}_{3L} + \beta T^{db}_{8L} + N_1\delta^{db}\right)\right] \\
&\approx \delta^{ac}u^c v^d \delta^{db} + ie\left(\delta^{ac}u^c v^d T^{db}_{3L} + \beta\delta^{ac}u^c v^d T^{db}_{8L} + N_2\delta^{ac}u^c v^d\delta^{db} + T^{ac}_{3L}u^c v^d \delta^{db}\right. \\
&\quad \left. + \beta T^{ac}_{8L}u^c v^d\delta^{db} + N_1\delta^{ac}u^c v^d\delta^{db}\right) \\
&= u^a v^b + ie\left(u^a v^d T^{db}_{3L} + T^{ac}_{3L}u^c v^b\right) + ie\beta\left(u^a v^d T^{db}_{8L} + T^{ac}_{8L}u^c v^b\right) + ie\left(N_1 + N_2\right)u^a v^b \\
&= \Lambda^{ab} + ie\{\Lambda, T_{3L} + \beta T_{8L}\}^{ab} + ieN\Lambda^{ab},
\end{aligned}
\tag{2.25}
$$

where $N = N_1 + N_2$. Therefore,

$$
\begin{aligned}
S' &= \frac{\Lambda + \Lambda^T}{2} + ie\left\{\frac{\Lambda + \Lambda^T}{2}, T_{3L} + \beta T_{8L}\right\} + ieN\frac{\Lambda + \Lambda^T}{2} \\
&= S + ie\{S, T_{3L} + \beta T_{8L}\} + ieNS
\end{aligned}
$$

So, the charge operator for the sextet is

$$
Q(S) = \left\{T^3_L + \beta T^8_L, S\right\} + NS. \tag{2.26}
$$

It is interesting to observe that the charge operator takes the form of an anticommutator when it is built from the multiplication of two triplets. It can be verified that it would have the form of a scalar operator if the field had been built from the multiplication of a triplet with an antitriplet.

This process also deduces the way in which the sextet is transformed, namely $S' = USU^T$ as can be observed in the first line of the equation (2.25). This deduction also show how the sextet gets its hypercharge from hypercharges of constituent triplets $N = N_1 + N_2$. In the model used, to be in accordance with (2.6), $N_1 = N_2$.

Then, the electric charge of each sextet components will be

$$
\begin{aligned}
Q(S) &= \begin{pmatrix} 1+\frac{\beta}{\sqrt{3}}+N & \frac{\beta}{\sqrt{3}}+N & \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}\right)+N \\ \frac{\beta}{\sqrt{3}}+N & -1+\frac{\beta}{\sqrt{3}}+N & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}\right)+N \\ \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}\right)+N & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}\right)+N & -\frac{2\beta}{\sqrt{3}}+N \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3}(1-q)+N & -\frac{1}{3}(1+2q)+N & \frac{1}{3}(2+q)+N \\ -\frac{1}{3}(1+2q)+N & -\frac{2}{3}(2+q)+N & -\frac{1}{3}(1-q)+N \\ \frac{1}{3}(2+q)+N & -\frac{1}{3}(1-q)+N & \frac{2}{3}(1+2q)+N \end{pmatrix}
\end{aligned} \tag{2.27}
$$

For the possible values of β and $S = \eta\eta^T$ $(N(S) = 2N(\eta))$,

$$
\begin{aligned}
&\beta = -\sqrt{3} \quad \text{or} \quad q = +1, \quad N(S) = 0: \qquad Q(S) = \begin{pmatrix} 0 & -1 & +1 \\ -1 & -2 & 0 \\ +1 & 0 & +2 \end{pmatrix}, \\
&\beta = +\sqrt{3} \quad \text{or} \quad q = -2, \quad N(S) = -2: \quad Q(S) = \begin{pmatrix} 0 & -1 & -2 \\ -1 & -2 & -3 \\ -2 & -3 & -4 \end{pmatrix}, \\
&\beta = -\tfrac{1}{\sqrt{3}} \quad \text{or} \quad q = 0, \quad N(S) = -\tfrac{2}{3}: \quad Q(S) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \\
&\beta = \tfrac{1}{\sqrt{3}} \quad \text{or} \quad q = -1, \quad N(S) = -\tfrac{4}{3}: \quad Q(S) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -2 \end{pmatrix}.
\end{aligned} \tag{2.28}
$$

For each β there is a different number of neutral scalars (Higgs) responsible for assigning Majorana mass terms to neutrinos, which reinforce the idea that the whole theory is β-value dependent.

2.5 Octet

In the same foot as the SM quartet, there is the possibility of include an octet accepted by the group representation in $SU(3)$. One way to obtain this multiplet is by multiplying $3 \otimes 3^* = 1 \oplus 8$.

In tensor form, $u^a v_b = \left(u^a v_b - \frac{1}{3}\delta^a_b u^k v_k\right) + \frac{1}{3}\delta^a_b u^k v_k$ being u^a, v_b triplet and anti-triplet respectively. The first addend $\Omega^a_b = u^a v_b - \frac{1}{3}\delta^a_b u^k v_k$ is the searched octet which transforms as

$$\begin{aligned} \Omega' = U u v^\dagger U^\dagger &\approx [\mathbb{1} + ie(T_{3L} + \beta T_{8L}) + N_1]\,\Omega\,[\mathbb{1} - ie(T_{3L} + \beta T_{8L}) + N_2] \\ &\approx \Omega + ie[\Omega, \mathbb{1} + ie(T_{3L} + \beta T_{8L})] + ieN\Omega, \end{aligned} \tag{2.29}$$

where $N = N_1 - N_2$. If the octet is built from multiplets with the same hypercharges, then $N = 0$. In this case, $Q\Omega = [\Omega, \mathbb{1} + ie(T_{3L} + \beta T_{8L})]$ or

$$Q(\Omega) = \begin{pmatrix} 0 & +1 & \frac{1}{2}(1+\sqrt{3}\beta) \\ -1 & 0 & \frac{1}{2}(-1+\sqrt{3}\beta) \\ -\frac{1}{2}(1+\sqrt{3}\beta) & \frac{1}{2}(1-\sqrt{3}\beta) & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 & -q \\ -1 & 0 & -q-1 \\ +q & +q+1 & 0 \end{pmatrix}. \tag{2.30}$$

For the possible β values,

$$\begin{aligned} \beta = -\sqrt{3} \quad \text{or} \quad q = +1: \quad Q(\Omega) &= \begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & -2 \\ +1 & +2 & 0 \end{pmatrix}, \\ \beta = +\sqrt{3} \quad \text{or} \quad q = -2: \quad Q(\Omega) &= \begin{pmatrix} 0 & +1 & +2 \\ -1 & 0 & +1 \\ -2 & -1 & 0 \end{pmatrix}, \\ \beta = -\frac{1}{\sqrt{3}} \quad \text{or} \quad q = 0: \quad Q(\Omega) &= \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \\ \beta = \frac{1}{\sqrt{3}} \quad \text{or} \quad q = -1: \quad Q(\Omega) &= \begin{pmatrix} 0 & +1 & +1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.31}$$

The last case, with $\beta = \frac{1}{\sqrt{3}}$ match with the known electric charge for the pseudoscalar meson octet matrix.

Chapter 3

Modelo $\mathbf{SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}}$

In the SM there is an inherent left-right disparity since the right and left multiplets are not treated in the same footing. Is that a characteristic of nature? There is a possibility whereby left-right symmetry is recovered at high energies and this is achieved in L-R symmetric models.

As mentioned in the introduction, this extension maintains a parity symmetry [42, 43] which breaks spontaneously to the SM symmetry at very high energies and then breaks again to $U(1)_Q$ at electroweak energy scale (~ 1 TeV). In order to do that, new scalars are needed with their respective vacuum expectation values, such as Higgs triplets and bi-doublets which permit the symmetry to be reduced accordingly. Also, we are considering the model that seesaw mechanism is allowed.

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \longrightarrow SU(2)_L \otimes U(1)_Y \longrightarrow U(1)_Q \tag{3.1}$$

However, there is some research in recent literature that uses models considering the possibility that there is no parity restoration at high energies but it is broken explicitly and the interactions of left- and right- handed fermions are completely different from each other at any energy scale [44].

The old and more used model described here, in addition to establishing a relationship between the restoration of parity at high energies (GUT energy scales) and the seesaw mechanism [45], try to explain interactions that violate CP [46].

We will consider the so-called Minimal Left-Right Symmetric Model (MLRSM) with a bi-doublet and a triplet scalar, where the parity invariance is imposed before the SSB down to SM symmetry.

3.1 Lagrangian

The electroweak Lagrangian in this symmetry is similar to that of SM in eq. (1.1) but includes new multiplets

$$
\begin{aligned}
\mathscr{L} \; &= -\tfrac{1}{4}F_{iL,R\,\mu\nu}F^{\mu\nu}_{iL,R} - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} && \text{...Vector sector}\\
&+i\overline{\Psi}_{iL,R}\not{D}\Psi_{iL,R} + h.c. && \text{...Fermion-Vector sector}\\
&+\overline{\Psi}_{iL}\left(D_{ij}\Phi + D'_{ij}\widetilde{\Phi}\right)\Psi_{jR} + h.c. && \\
&+i\,\overline{(L_{iL,R})^c}M_{ij}\sigma_2\Delta_{L,R}L_{jL,R} + h.c. && \text{...Interaction sector (Yukawa)}\\
&+\mathrm{Tr}\left[(D_\mu\Delta_{L,R})^\dagger(D^\mu\Delta_{L,R})\right] + \mathrm{Tr}\left[(D_\mu\Phi)^\dagger(D^\mu\Phi)\right] - V(\Phi) && \text{...Scalar-Vector sector (Higgs)}
\end{aligned}
\tag{3.2}
$$

The relevant aspects we have to point out about the terms in this Lagrangian are:

- In the vector sector there are six tensors for the $SU(2)_L \otimes SU(2)_R$ symmetry, three for $F_{iL\,\mu\nu} = \partial_\mu W_{iL\,\nu} - \partial_\nu W_{iL\,\mu} + g_L\epsilon_{ijk}W_{jL\,\mu}W_{kL\nu}$, $i = 1\ldots3$, three for $F_{iR\,\mu\nu} = \partial_\mu W_{iR\,\nu} - \partial_\nu W_{iR\,\mu} + g_R\epsilon_{ijk}W_{jR\,\mu}W_{kR\nu}$ whose gauge vector fields $W_{iL,R\,\mu}$ are associated to $SU(2)_{L,R}$ generators (ϵ_{ijk} are the structure constant of this Lie group) and one tensor for the $U(1)_{B-L}$ symmetry $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ where B_μ is associated to the single generator.

- In the fermion sector left doublets Ψ_{iL}, $i = 1\ldots3$ are representations of the left-handed leptons L_{iL} or quarks Q_{iL}, while Ψ_{iR} are the corresponding right-handed doublets L_{iR} or Q_{iR}.

- The Yukawa sector takes into account two classes of scalar multiplets, the first one are the bi-doublets Φ and $\widetilde{\Phi} = \sigma_2\Phi\sigma_2$ which couples with left anti-doublets and right doublets like in $\overline{Q}_{iL}\Phi Q_{jR}$ (that is why it must be a bi-doublet). The need use $\widetilde{\Phi}$ in addition to Φ is to obtain all possible couplings with fermions. The second class is the triplet $\Delta_{L,R}$ which ensures Majorana mass terms for neutrinos to perform seesaw mechanism [47].

- In the Higgs sector, since scalar multiplets are bi-doublets and triplets (the latter written in its adjoint representation), their corresponding kinetic terms are written using a Hilbert-Schmidt inner product, $\mathrm{Tr}\left[(D^\mu\Phi)^\dagger(D_\mu\Phi)\right]$ or $\mathrm{Tr}\left[(D^\mu\Delta_{L,R})^\dagger(D_\mu\Delta_{L,R})\right]$. The potential $V(\Phi)$

is defined in its most possible general form considering these scalars and the discrete symmetries if aplicable to the theory [48].

The covariant derivatives for fermion doublets $\Psi_{iL,R}$ are

$$\begin{aligned} D^\mu \Psi_{iL} &= \left[\partial^\mu + i g_L T_{jL} W^\mu_{jL} + i g_{B-L} B^\mu Y_{B-L}\right] \Psi_{iL}, \\ D^\mu \Psi_{iR} &= \left[\partial^\mu + i g_R T_{jR} W^\mu_{jR} + i g_{B-L} B^\mu Y_{B-L}\right] \Psi_{iR}, \end{aligned} \tag{3.3}$$

where:

- g_L, g_R y g_{B-L} are the coupling constants of $SU(2)_L$, $SU(2)_R$ and $U(1)_{B-L}$ respectively.
- Values $Y_{B-L} = B - L$ are the corresponding hypercharges.
- $W^\mu_{jL,R}$ with $j = 1, 2, 3$ are the six gauge fields that correspond to $SU(2)_{L,R}$ generators and B^μ is the gauge field of $U(1)_{B-L}$ group. It is important to remark that after breaking these gauge bosons mix themselves to form the physical (mass) states $W^{\mu\pm}_L$ and $W^{\mu\pm}_R$, Z^μ_L and Z^μ_R as well as A^μ. However, $(W, Z)_L$ and $(W, Z)_R$ have very different masses after breaking.

Left- and right-handed fermions doublets transform as

$$\begin{aligned} \Psi_{iL} &\to \Psi'_{iL} = U_{B-L} U_L \Psi_{iL}, \\ \Psi_{iR} &\to \Psi'_{iR} = U_{B-L} U_R \Psi_{iR}, \end{aligned} \tag{3.4}$$

where $U_{L,R} = \exp\left[i g_{L,R} T_j \omega_{jL,R}(x)\right]$ and $U_{B-L} = \exp\left[i g_{B-L} Y_{B-L} f(x)\right]$. Like in SM, $T_j = \frac{\sigma_j}{2}$ are the $SU(2)_{L,R}$ generators and $Y_{B-L} = B - L$ is the hypercharge.

The gauge fields transform as

$$\begin{aligned} W^\mu_{iL,R}(x) &\to W^{\mu'}_{iL,R}(x) = W^\mu_{iL,R}(x) - \partial^\mu \omega_{iL,R}(x) - g_{L,R} \epsilon_{ijk} \omega_{jL,R}(x) W^\mu_{kL,R}(x), \\ B^\mu(x) &\to B^{\mu'}(x) = B^\mu(x) - \partial^\mu f(x), \end{aligned} \tag{3.5}$$

where $f(x)$ and $\omega_{jL,R}(x)$ are real function-like parameters and $\epsilon_{ijk} = \frac{1}{4i} \operatorname{Tr}\{[\sigma_i, \sigma_j]\, \sigma_k\}$ is the Levi-Civita tensor or the structure constant defined for $SU(2)_{L,R}$ group. Matrices σ_i are the Pauli matrices.

The covariant derivatives for scalar multiplets are

$$
\begin{aligned}
D^{\mu}\Phi &= \partial^{\mu}\Phi + i\left[g_L T_j W^{\mu}_{jL}\Phi - g_R \Phi T_j W^{\mu}_{jR}\right], \\
D^{\mu}\Delta_{L,R} &= \partial^{\mu}\Delta_{L,R} + i g_{L,R}\left[T_j W^{\mu}_{jL,R}, \Delta_{L,R}\right] + i g_{B-L} B^{\mu}\Delta_{L,R}.
\end{aligned} \tag{3.6}
$$

For the $\Delta_{L,R}$ case, its derivative has the usual form with a commutator due to the way they are constructed (See (F.13)). But, for the Φ case, its derivative has a special treatment as we will show later.

In accordance with the Yukawa sector in Lagrangian (3.2),

$$
\begin{aligned}
\overline{\Psi_{iL}}\Phi\Psi_{jR} &: 2^*_L \otimes \Phi \otimes 2_R = 1 \Rightarrow \Phi = 2_L \otimes 2^*_R \\
\overline{(\Psi_{iL,R})^c}(i\sigma_2)\Delta_{L,R}\Psi_{jL,R} &: (2^* \otimes \Delta \otimes 2)_{L,R} = 1 \Rightarrow \Delta_{L,R} = (2 \otimes 2^*)_{L,R} = (3 \oplus 1)_{L,R}.
\end{aligned}
$$

Then, Φ transforms as a bi-doublet whilst $\Delta_{L,R}$ as a triplet in its adjoint representation to be coupled accordingly to fermions. Like any symmetry triplet $\Delta_{L,R} \rightarrow \Delta'_{L,R} = U_{B-L}U_{L,R}\Delta_{L,R}$, being $U_{L,R}$ the spin-1 representation from $SU(2)$, that is the reason why it is written in the Lagrangian in its adjoint representation. The gauge transformations are

$$
\begin{aligned}
\Phi &\rightarrow \Phi' = U_L \Phi U_R^{\dagger}, \\
\Delta_{L,R} &\rightarrow \Delta'_{L,R} = U_{B-L}U_{L,R}\Delta_{L,R}U^{\dagger}_{L,R}.
\end{aligned} \tag{3.7}
$$

These $\Delta_{L,R}$ transformations allow terms such as $\overline{(L_{iL,R})^c} h^M_{ij}\Delta_{L,R}L_{jL,R}$ to give Majorana mass terms for neutrinos after SSB, while terms including Φ or $\widetilde{\Phi}$ permit ad-hoc Dirac mass terms for all fermions like $\overline{L_{iL,R}}(h_{ij}\Phi + \widetilde{h}_{ij}\widetilde{\Phi})L_{jL,R}$.

In the next section, the particle content is described separately in the fermionic, bosonic and scalar sector of the lagrangian; all of them in terms of their electric charge operators.

3.2 Conserved charges and particle content

The charge relation is,

$$
\frac{Q}{e} = T_{3L} + T_{3R} + \left(\frac{B-L}{2}\right)\mathbb{1}_2. \tag{3.8}
$$

As in SM, T_{3L} and T_{3R} are the third isospin generators, but in this case they act only on the left (T_{3L}) or right (T_{3R}) multiplets respectively. The letters B and L are the baryonic and leptonic number respectively. Besides, the introduction of the $B-L$ symmetry contains the existence not only of Majorana neutrinos ($\Delta L = 2$) but also $n \leftrightarrow \bar{n}$ transitions ($\Delta B = 2$) when breaking this symmetry [49].

3.2.1 Fermions

This symmetry requires the introduction of right "partners" for the fermions and gauge bosons. Unlike SM, where the left fermionic fields are presented in doublets and the right ones in singlets of the symmetry, here all fermions (left and right) have a doublet representation for the three flavour families $i = 1, 2, 3$ [43].

$$\Psi_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \sim (\mathbf{2_L}, \mathbf{1_R}, B-L)\,;\; \Psi_{iR} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_R \sim (\mathbf{1_L}, \mathbf{2_R}, B-L)\,, \tag{3.9}$$

where the hypercharge is $B - L$[1]. This choice of hypercharge ensures the known electrical charge of the components that are assigned after the SSB with Eq. (3.8).

In SM the procedure (1.5) was done for doublets. Here this procedure is done twice (for the right and left representations) and the electric charge operator for fermions are:

$$Q(\Psi_{L,R}) = \frac{1}{2}\begin{pmatrix} 1+B-L & 0 \\ 0 & -1+B-L \end{pmatrix}_{L,R}. \tag{3.10}$$

In lepton sector, doublets have $B = 0$ and $L = 1$ while in quark sector $B = 1/3$ and $L = 0$. That assignment gives the expected values for electric charges.

1st	2nd	3rd	$\mathrm{T_{3L,R}}$	Q	B − L
$\begin{pmatrix} u \\ d \end{pmatrix}_{L,R}$	$\begin{pmatrix} c \\ s \end{pmatrix}_{L,R}$	$\begin{pmatrix} t \\ b \end{pmatrix}_{L,R}$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}_{L,R}$	$\begin{pmatrix} +\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}_{L,R}$	$+\frac{1}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,R}$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_{L,R}$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_{L,R}$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}_{L,R}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}_{L,R}$	-1

Table 3.1: Fermionic charges in 3221-LR families.

[1]The factor $1/2$ is a matter of convention like in the SM to get integers numbers mostly.

3.2.2 Vector Bosons

Like in the SM, before the breaking gauge bosons within the Lagrangian are represented as triplets in the adjoint representation of $SU(2)_L : \widetilde{W}_{\mu L} \sim (\mathbf{3_L}, \mathbf{1_R})$ and of $SU(2)_R : \widetilde{W}_{\mu R} \sim (\mathbf{1_L}, \mathbf{3_R})$.

$$Q(\widetilde{W}_{\mu L,R}) = \left[T^3, \widetilde{W}_\mu\right]_{L,R} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}_{L,R} . \tag{3.11}$$

However, it should be noted that, although the electrical charges of the left components are the same as those of their right counterparts, the masses after breaking are very different. Unlike fermions, the denomination left or right do not incise in the mass value; they are only labels to indicate belonging to a certain symmetry (L or R) and to identify which boson comes out of the covariant derivative that acts on the left or right doublets. The mass of the right components turns out to be much greater than their left counterparts and that explain their existence at very high energies justifying the reason why they are not found yet.

The gauge-covariant derivative is

$$D^\mu = \partial^\mu + ig_L W^\mu_{iL} T_{iL} + ig_R W^\mu_{iR} T_{iR} - ig_{BL} \frac{(B-L)}{2} B_\mu . \tag{3.12}$$

The matrices $T_{iL,R}$ are the three $SU(2)_{L,R}$ generators and act only to the L, R doublets respectively.

It is common to make in the Minimal L-R Symmetric Model (MLRSM) $g_L = g_R = g$ if you want the theory to have parity symmetry from the beginning. Considering that, we may define two phases that relate the coupling constants $\cos\theta = \sqrt{\frac{g^2 + g_{BL}}{g^2 + 2g_{BL}}}$ and $\cos\theta_{BL} = \sqrt{\frac{g^2}{g^2 + g_{BL}}}$. It can be demonstrated (See appendix (E), eq. (E.8)) that

$$\frac{1}{e^2} = \frac{2}{g^2} + \frac{1}{g^2_{BL}} . \tag{3.13}$$

In order to be in accordance with the fact that this symmetry contains that of SM, the left handed contributions have to coincide with the SM couplings. That means the g is assumed to have the same value as the SM g_L, then $g_{BL} = 0.4133$.

3.2.3 Scalar Bosons

Following the MLRSM presented in [43], the scalar multiplets can are represented with one bi-doublet and two triplets [43]. The bi-doublet Φ gives Dirac mass terms to fermions while the triplets give Majorana mass terms to neutrinos.

$$\begin{aligned} \Phi &= \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \sim (\mathbf{2}_L, \mathbf{2}^*_R, 0)\,, \\ \Delta_L &= \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}_L \sim (\mathbf{3}_L, \mathbf{1}_R, +2)\,, \\ \Delta_R &= \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}_R \sim (\mathbf{1}_L, \mathbf{3}_R, +2)\,, \end{aligned} \tag{3.14}$$

Bidoublets have the representation $2_L \otimes 2^*_R$ and may be built from $\Phi = \begin{pmatrix} \phi_1 & \phi_2^c \end{pmatrix}$, where $\phi_{1,2}$ are left Higgs doublets and then ϕ_2^c is right handed. Since the two doublets are opposite in all charges, hypercharge $B-L$ for Φ is always zero and it is suitable for coupling with fermions forming Dirac mass terms after breaking. Then,

$$\begin{aligned} \Phi \to \Phi' &= U_L \Phi U_R^\dagger \\ &\approx (\mathbf{1}_2 + ieT_{3L})\,\Phi\,(\mathbf{1}_2 - ieT_{3R}) \\ &= \Phi + ie\,(T_{3L}\Phi - \Phi T_{3R})\,. \end{aligned}$$

The charge operator for bidoublet is

$$Q(\Phi) = T_{3L}\Phi - \Phi T_{3R} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{3.15}$$

which means that the bidoublet components may be written down as

$$\Phi = \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & -\phi_2^{0*} \end{pmatrix}. \tag{3.16}$$

For the triplets case, in $SU(2)_{L,R}$, it is verified that $2 \otimes 2^* = 3 \oplus 1$ [41]. In tensor form, be u^a, v_b, $a, b = 1, 2$ are doublet and anti-doublet of $SU(2)$ respectively, then their multiplication may be represented in matrix form,

$$\begin{aligned}
\mathcal{T}^a_b = u^a v_b &= \begin{pmatrix} u^1 v_1 & u^1 v_2 \\ u^2 v_1 & u^2 v_2 \end{pmatrix} \\
&= \begin{pmatrix} u^1 v_1 & u^1 v_2 \\ u^2 v_1 & u^2 v_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} u^1 v_1 + u^2 v_2 & 0 \\ 0 & u^1 v_1 + u^2 v_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} u^1 v_1 + u^2 v_2 & 0 \\ 0 & u^1 v_1 + u^2 v_2 \end{pmatrix} \\
&= \left(u^a v_b - \frac{1}{2}\delta^a_b u^k v_k \right) + \frac{1}{2}\delta^a_b u^k v_k .
\end{aligned}$$

The first addend has the shape or a $SU(2)$ triplet tensor, then it could be related with $\Delta_{L,R} \sim 3_{L,R}$. The second addend is a singlet $\sim 1_{L,R}$. Since doublets transform as $u_{L,R} \to u'_{L,R} = U_{B-L} U_{L,R} u_{L,R}$ then the triplet $\Delta_{L,R}$ is transformed as

$$\begin{aligned}
\Delta'_{L,R} &= \left(U_{B-L} U u\right)_{L,R} \left(v^c U^*_{B-L} U^\dagger\right)_{L,R} \\
&\approx \left[\mathbf{1}_2 + ie\left(T_{3L,R} + \left(\frac{B-L}{2}\right)_1 \mathbf{1}_2\right)\right] (uv^c) \left[\mathbf{1}_2 - ie\left(T_{3L,R} + \left(\frac{B-L}{2}\right)^*_2 \mathbf{1}_2\right)\right] \\
&\approx uv^c + ie\left(T_{3L,R} uv^c - uv^c T_{3L,R}\right) + ie\left(\left(\frac{B-L}{2}\right)_1 - \left(\frac{B-L}{2}\right)^*_2\right) uv^c \\
&= \Delta_{L,R} + ie[T_{3L,R}, \Delta_{L,R}] + \left(\frac{B-L}{2}\right)\Delta_{L,R},
\end{aligned} \tag{3.17}$$

where $B-L = (B-L)_1 - (B-L)^*_2$ and the subscripts $1,2$ correspond to the doublet hypercharges of u, v respectively. Therefore, the charge operator for the triplet is

$$Q(\Delta_{L,R}) = [T_{3\,L,R}, \Delta_{L,R}] + \left(\frac{B-L}{2}\right)\Delta_{L,R}. \tag{3.18}$$

The electric charge operator is

$$Q(\Delta_{L,R}) = \frac{1}{2}\begin{pmatrix} B-L & +2+B-L \\ -2+B-L & B-L \end{pmatrix}_{L,R}. \tag{3.19}$$

It can be noticed that if we choose $u = v$, then $B-L$ must be twice the hypercharge of u. This makes sense because $\Delta_{L,R}$ is coupled with two doublets with same hypercharge in the lepton sector $L_{L,R}$ and $\overline{(L_{L,R})^c}$. Moreover, in this model these doublets have hypercharge -1 and so the hypercharge of $\Delta_{L,R}$ must be $+2$. Therefore, the electric charge operator is

$$Q(\Delta_{L,R}) = \begin{pmatrix} +1 & +2 \\ 0 & +1 \end{pmatrix}_{L,R}, \tag{3.20}$$

and the triplet content may be written as

$$\Delta_{L,R} = \begin{pmatrix} \frac{\delta^+}{\sqrt{2}} & \delta^{++} \\ \delta^0 & -\frac{\delta^+}{\sqrt{2}} \end{pmatrix}_{L,R}, \tag{3.21}$$

which came from the original triplet gauge states δ_i, $i = 1, 2, 3$ before breaking.

Regarding to exotic multiplets, we can replicate the same SM multiplets, like in (1.4.4), but with left and right chiralities.

Chapter 4

Charges in the model $\mathbf{SU(3)_C \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_X}$

The development of left-right symmetry extensions to the SM were motivated by the explanation of the parity (and/or charge conjugation) violation in weak interactions observed at low energy. In this case, the model carries the benefits of the 331 and 221-LR model: the latter in which parity symmetry is preserved and the former which explains why only three families are observed by anomaly cancellation (see the appendix in [9]).

The chosen model is able to generate Dirac or Majorana masses for neutrinos and is a left-right extension of the 331 chiral model explained before. The charge relation is

$$\mathcal{Q} = T_{3L} + T_{3R} + \beta \left(T_{8L} + T_{8R}\right) + X, \tag{4.1}$$

where, as in 221 L-R models, operators $T_{3L,R}$ y $T_{8L,R}$ are applied over L or R fields only.

The particle content is in accordance with [9] and [10], but for a general β. However, it has to be mentioned that this symmetry was considered firstly in [12] considering only scalar triplets and that is why fermion masses were generated by five-dimensional operators.

4.1 Lagrangian

The electroweak Lagrangian in this symmetry is similar to the previous ones with extended multiplets.

$$
\begin{aligned}
\mathscr{L} \;&= -\tfrac{1}{4}F_{iL,R\,\mu\nu}F^{\mu\nu}_{iL,R} - \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} && \ldots \text{Vector sector} \\
&+i\overline{\Psi}_{lL,R}\not{D}\Psi_{lL,R} + h.c. && \ldots \text{Fermion-Vector sector} \\
&+\overline{Q}_{aL}G_{ab}\Phi^{*}Q_{bR} + \overline{Q}_{3L}G_{33}\Phi Q_{3R} + \overline{Q}_{aL}G_{a3}P^{*}Q_{3R} + \overline{Q}_{3L}G_{3a}PQ_{aR} && \text{Interaction sector} \\
&+\overline{L}_{l}G_{lm}\Phi L_{m} + G'_{lm}\left(\overline{L^{c}_{l}}\Delta^{\dagger}_{L}L_{m} + \overline{R^{c}_{l}}\Delta^{\dagger}_{R}R_{m}\right) + h.c. && \ldots \text{(Yukawa)} \\
&+\mathrm{Tr}\left[(D_{\mu}\Delta_{L,R})^{\dagger}(D^{\mu}\Delta_{L,R})\right] + \mathrm{Tr}\left[(D_{\mu}\Phi)^{\dagger}(D^{\mu}\Phi)\right] && \text{Scalar-Vector} \\
&+\mathrm{Tr}\left[(D_{\mu}P)^{\dagger}(D^{\mu}P)\right] - V(\Phi,\Delta,P) && \ldots \text{sector (Higgs)}
\end{aligned}
\tag{4.2}
$$

The principal features of this Lagrangian are:

- In the vector sector there are sixteen tensors for the $SU(3)_L \otimes SU(3)_R$ symmetry, eight for $F_{iL\,\mu\nu} = \partial_\mu W_{iL\,\nu} - \partial_\nu W_{iL\,\mu} + g_L f_{ijk} W_{jL\,\mu} W_{kL\nu}$, $i,j,k = 1\ldots 8$, eight for $F_{iR\,\mu\nu} = \partial_\mu W_{iR\,\nu} - \partial_\nu W_{iR\,\mu} + g_R f_{ijk} W_{jR\,\mu} W_{kR\nu}$ whose gauge fields $W_{iL,R\,\mu}$ are associated to $SU(3)_{L,R}$ generators (f_{ijk} are the structure constant of this Lie group) and one tensor for the $U(1)_X$ symmetry $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ where B_μ is associated to the single generator.

- In the fermion sector Ψ_{lL} are left triplets, where $l = 1\ldots 3$ are flavour index. Representations of the left-handed leptons are written as L_{lL} and quarks as Q_{lL}, while Ψ_{lR} are the corresponding right-handed triplets L_{lR} or Q_{lR}.

- The Yukawa sector was written down taking into account after breakdown Dirac mass terms for all the fermions and Majorana mass terms for neutrinos. Then, the following must be fulfilled for the scalar multiplets Φ, $\Delta_{L,R}$ and P:

$$
\begin{aligned}
\overline{Q_{3L}}\,\Phi\,Q_{3R} &: 3^{*}_{L} \otimes \Phi \otimes 3_{R} = 1 \Rightarrow \Phi = 3_{L} \otimes 3^{*}_{R} \\
\overline{(L,R)^{c}_{l}}\,\Delta^{\dagger}_{L,R}(L,R)_{m} &: 3_{L,R} \otimes \Delta^{\dagger}_{L,R} \otimes 3_{L,R} \Rightarrow \Delta_{L,R} = 3_{L,R} \otimes 3_{L,R} = (3^{*} \oplus 6)_{L,R} \\
\overline{Q_{3L}}\,P\,Q_{aR} &: 3^{*}_{L} \otimes P \otimes 3^{*}_{R} = 1 \Rightarrow P = 3_{L} \otimes 3_{R}
\end{aligned}
$$

  which means that the usual Dirac mass terms are obtained if Φ is a bi-triplet and P is a bi-fundamental multiplet while the Majorana mass terms are achieved with $\Delta_{L,R}$ being a left or right sextet. The bi-triplet Φ couples with left and right quark triplets belonging only to the third family due to their hypercharge assignment (as we will see later), that is why $a, b = 1, 2$. The sextets $\Delta_{L,R}$ ensures Majorana mass terms for neutrinos to perform seesaw mechanism and the bi-fundamental field P completes the coupling with third family of quarks.

- In the Higgs sector, since scalar multiplets are bi-triplets, bi-fundamentals and sextets, their corresponding kinetic terms are written using a Hilbert-Schmidt inner product, $\mathrm{Tr}\left[(D^\mu\Phi)^\dagger (D_\mu\Phi)\right]$, $\mathrm{Tr}\left[(D^\mu\Delta_{L,R})^\dagger (D_\mu\Delta_{L,R})\right]$ and $\mathrm{Tr}\left[(D^\mu P)^\dagger (D_\mu P)\right]$. As usual, the potential $V(\Phi, \Delta, P)$ is defined in its most possible general form considering discrete symmetries if aplicable to the theory [9].

The covariant derivatives for fermion triplets $\Psi_{iL,R}$ are

$$\begin{aligned} D^\mu\Psi_{iL} &= \left[\partial^\mu + ig_L T_{jL} W^\mu_{jL} + ig_X B^\mu X\right]\Psi_{iL}, \\ D^\mu\Psi_{iR} &= \left[\partial^\mu + ig_R T_{jR} W^\mu_{jR} + ig_X B^\mu X\right]\Psi_{iR}, \end{aligned} \tag{4.3}$$

where:

- g_L, g_R y g_X are the coupling constants of $SU(3)_L$, $SU(3)_R$ and $U(1)_X$ respectively.

- X is the corresponding hypercharge.

- $W^\mu_{jL,R}$ with $j = 1 \ldots 8$ are sixteen gauge fields that correspond to $SU(3)_{L,R}$ generators that mix with B^μ, the gauge field of $U(1)_X$, to form mass (physical) states after breaking like in SM. It is important to remark that W^μ_L and W^μ_R are not related after SSB.

Fundamental representations (fermions triplets) transform as

$$\begin{aligned} \Psi_{lL} &\to \Psi'_{lL} = U_X U_L \Psi_{lL}, \\ \Psi_{lR} &\to \Psi'_{lR} = U_X U_R \Psi_{lR}, \end{aligned} \tag{4.4}$$

where $U_{L,R} = \exp\left[ig_{L,R} T_j \omega_{jL,R}(x)\right]$ and $U_X = \exp\left[ig_X X f(x)\right]$. Like in 331, $T_j = \frac{\lambda_j}{2}$ are the $SU(3)_{L,R}$ generators and X is the hypercharge.

The gauge fields transform as

$$\begin{aligned} W^\mu_{iL,R}(x) &\to W^{\mu'}_{iL,R}(x) = W^\mu_{iL,R}(x) - \partial^\mu\omega_{iL,R}(x) - g_{L,R} f_{ijk}\omega_{jL,R}(x) W^\mu_{kL,R}(x), \\ B^\mu(x) &\to B^{\mu'}(x) = B^\mu(x) - \partial^\mu f(x), \end{aligned} \tag{4.5}$$

where $f(x)$ and $\omega_{iL,R}(x)$ are real function-like parameters and $f_{ijk} = \frac{1}{4i}\mathrm{Tr}\{[\lambda_i, \lambda_j]\,\lambda_k\}$ are the structure constants defined for $SU(3)_{L,R}$ group. Matrices λ_i are the Gell-Mann matrices.

The covariant derivatives for scalar multiplets are (see appendix (F))

$$\begin{aligned} D^\mu\Phi &= \partial^\mu\Phi + i\left[g_L T_j W^\mu_{jL}\Phi - g_R\Phi T_j W^\mu_{jR}\right], \\ D^\mu\Delta_{L,R} &= \partial^\mu\Delta_{L,R} + ig_{L,R}\left[T_j W^\mu_{jL,R}\Delta_{L,R} + \Delta_{L,R}(T_j W^\mu_{jL,R})^T\right] + ig_X X B^\mu\Delta_{L,R}, \\ D^\mu P &= \partial^\mu P + i\left[g_L T_j W^\mu_{jL} P + g_R P(T_j W^\mu_{jR})^T\right] + ig_X X B^\mu P. \end{aligned} \tag{4.6}$$

As in the chiral 331 case, the sextets have a particular way of transforming without a commutator. In the three cases, the scalar multiplet transformations lead us to write the way they would be built accordingly.

$$\begin{aligned} \Phi &\to \Phi' = U_L\Phi U_R^\dagger, \\ \Delta_{L,R} &\to \Delta'_{L,R} = U_X U_{L,R}\Delta_{L,R}U^T_{L,R}, \\ P &\to P' = U_X U_L P U_R. \end{aligned} \tag{4.7}$$

With all these transformations, the particle content can be described separately in the fermionic, bosonic and scalar sector of the lagrangian; all of them in terms of their electric charge operator.

4.2 Fermions

Quarks and leptons are left- and right-handed triplets (or antitriplets) like those of 331 case, and their charge operators are defined in equation (2.12),

$$\begin{aligned} Q(\mathcal{T}_{L,R}) &= \frac{1}{2}\mathrm{diag}\left(1+\frac{\beta}{\sqrt{3}}+2X, -1+\frac{\beta}{\sqrt{3}}+2X, -2\frac{\beta}{\sqrt{3}}+2X\right) \\ &= \mathrm{diag}\left(\frac{1-q}{3}+X, \frac{-2-q}{3}+X, \frac{2q+1}{3}+X\right) \\ Q(\mathcal{T}^*_{L,R}) &= \frac{1}{2}\mathrm{diag}\left(-1-\frac{\beta}{\sqrt{3}}+2N, 1-\frac{\beta}{\sqrt{3}}+2X^*, 2\frac{\beta}{\sqrt{3}}+2X^*\right) \\ &= \mathrm{diag}\left(\frac{-1+q}{3}+X^*, \frac{2+q}{3}+X^*, \frac{-2q-1}{3}+X^*\right). \end{aligned} \tag{4.8}$$

For simplicity, it is usual to write the components in terms of the electric charge of the third lepton component $q = \frac{-\sqrt{\beta}-1}{2}$.

$\beta = -\sqrt{3}$, $\mathbf{Q}$, $\mathbf{X}$	$\beta = +\sqrt{3}$	$\beta = -\frac{1}{\sqrt{3}}$	$\beta = \frac{1}{\sqrt{3}}$
$\mathcal{Q}_{3L,R},\begin{pmatrix}+2/3\\-1/3\\+5/3\end{pmatrix},+\frac{2}{3}$	$\begin{pmatrix}+2/3\\-1/3\\-4/3\end{pmatrix},-\frac{1}{3}$	$\begin{pmatrix}+2/3\\-1/3\\+2/3\end{pmatrix},+\frac{1}{3}$	$\begin{pmatrix}+2/3\\-1/3\\+1/3\end{pmatrix},0$
$\mathcal{Q}_{aL,R},\begin{pmatrix}-1/3\\+2/3\\-4/3\end{pmatrix},-\frac{1}{3}$	$\begin{pmatrix}-1/3\\+2/3\\+5/3\end{pmatrix},+\frac{2}{3}$	$\begin{pmatrix}-1/3\\+2/3\\-1/3\end{pmatrix},0$	$\begin{pmatrix}-1/3\\+2/3\\+2/3\end{pmatrix},+\frac{1}{3}$
$(L,R)_i,\begin{pmatrix}0\\-1\\+1\end{pmatrix},\ 0$	$\begin{pmatrix}0\\-1\\-2\end{pmatrix},-1$	$\begin{pmatrix}0\\-1\\0\end{pmatrix},-\frac{1}{3}$	$\begin{pmatrix}0\\-1\\-1\end{pmatrix},-\frac{2}{3}$

Table 4.1: Fermionic charges for different values of β and $a = 1, 2$.

These six triplets replicate the charges of the chiral 331 model, adding those of the right chirality.

4.3 Scalar bosons

Four scalar multiples are proposed: one bitriplet Φ, one bifundamental P and two symmetric sextets $\Delta_{L,R}$.

$$
\begin{aligned}
\Phi &= \begin{pmatrix}\eta_1 & \rho_1 & \chi_1\\ \eta_2 & \rho_2 & \chi_2\\ \eta_3 & \rho_3 & \chi_3\end{pmatrix} \sim (\mathbf{1}, \mathbf{3_L}, \mathbf{3^*_R}, 0),\ P = \begin{pmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{pmatrix} \sim (\mathbf{1}, \mathbf{3_L}, \mathbf{3_R}, X_P),\\
\Delta_L &= \begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\\ \sigma_2 & \sigma_4 & \sigma_5\\ \sigma_3 & \sigma_5 & \sigma_6\end{pmatrix}_L \sim (\mathbf{1}, \mathbf{6_L}, \mathbf{1_R}, X_{\delta_L}),\ \Delta_R = \begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\\ \sigma_2 & \sigma_4 & \sigma_5\\ \sigma_3 & \sigma_5 & \sigma_6\end{pmatrix}_R \sim (\mathbf{1}, \mathbf{1_L}, \mathbf{6_R}, X_{\delta_R}).
\end{aligned}
\tag{4.9}
$$

The bitriplet can be built from a left triplet and right antitriplet, $\Phi = v_L v_R^\dagger$, then the transformation is

$$
\begin{aligned}
\Phi \to \Phi' &= (U_X U_L v_L)\,(U_X U_R v_R)^\dagger = U_L \Phi U_R^\dagger\\
&\approx [\mathbf{1}_3 + ie\,(T_3 + \beta T_8)_L]\,\Phi\,[\mathbf{1}_3 - ie\,(T_3 + \beta T_8)_R]\\
&\approx \Phi + ie[T_3 + \beta T_8, \Phi],
\end{aligned}
$$

where the usual values of $\alpha_j(x)$ and $f(x)$ have been assumed like in chiral 331 for sextet S. The bitriplet has zero hypercharge as a consequence of its own transformation.

The charge operator for the bitriplet is

$$\begin{aligned} Q(\Phi) &= [T_3 + \beta T_8, \Phi] \\ &= \begin{pmatrix} 0 & +1 & \frac{1}{2}\left(1+\frac{3\beta}{\sqrt{3}}\right) \\ -1 & 0 & \frac{1}{2}\left(-1+\frac{3\beta}{\sqrt{3}}\right) \\ -\frac{1}{2}\left(1+\frac{3\beta}{\sqrt{3}}\right) & \frac{1}{2}\left(1-\frac{3\beta}{\sqrt{3}}\right) & 0 \end{pmatrix}. \end{aligned} \tag{4.10}$$

For the possible values of β,

$$\begin{aligned} &\beta = -\sqrt{3} : Q(\Phi) = \begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & -2 \\ +1 & +2 & 0 \end{pmatrix}, \ \beta = +\sqrt{3} : Q(\Phi) = \begin{pmatrix} 0 & +1 & +2 \\ -1 & 0 & +1 \\ -2 & -1 & 0 \end{pmatrix}, \\ &\beta = -\frac{1}{\sqrt{3}} : Q(\Phi) = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \ \beta = \frac{1}{\sqrt{3}} : Q(\Phi) = \begin{pmatrix} 0 & +1 & +1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.11}$$

This result is consistent with charges in table (2.4) and the fact that bitriplet is made up of $\Phi = (\eta, \rho, \chi)$ for any β.

The bifundamental multiplet can be built from a left triplet and right triplet, $P = w_L w_R^T$ with hypercharge X', then the transformation is

$$\begin{aligned} P \to P' &= (U_L w_L)(U_R w_R)^T = U_L P U_R^T \\ &\approx (1 + ieX') \left[\mathbf{1}_3 + ie\left(T_L^3 - \sqrt{3}T_L^8\right)\right] P (1 + ieX') \left[\mathbf{1}_3 + ie\left(T_R^3 - \sqrt{3}T_R^8\right)\right] \\ &\approx P + ie\{T^3 + \beta T^8, P\} + ieXP, \end{aligned}$$

where $X = 2X'$. The charge operator for P is

$$\begin{aligned} Q(P) &= \{T^3 + \beta T^8, P\} + XP = \\ &= \begin{pmatrix} \left(1+\frac{\beta}{\sqrt{3}}+X\right) & \left(\frac{\beta}{\sqrt{3}}+X\right) & \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}+2X\right) \\ \left(\frac{\beta}{\sqrt{3}}+X\right) & \left(-1+\frac{\beta}{\sqrt{3}}+X\right) & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}-2X\right) \\ \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}+2X\right) & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}-2X\right) & \left(-\frac{2\beta}{\sqrt{3}}+X\right) \end{pmatrix}. \end{aligned} \tag{4.12}$$

This result shows that this field can be considered as symmetric multiplet if the theory requires it.

P is design to be coupled with quarks, e.g. $\overline{Q}_{1L} P Q_{aR}$. Therefore, the hypercharge assignment depends on β.

$$\beta=-\sqrt{3}, X=+1 : Q(P)=\begin{pmatrix} +1 & 0 & +2 \\ 0 & -1 & +1 \\ +2 & +1 & +3 \end{pmatrix},$$

$$\beta=+\sqrt{3}, X=-1 : Q(P)=\begin{pmatrix} +1 & 0 & -1 \\ 0 & -1 & -2 \\ -1 & -2 & -3 \end{pmatrix},$$

$$\beta=-\frac{1}{\sqrt{3}}, X=+\frac{1}{3} : Q(P)=\begin{pmatrix} +1 & 0 & +1 \\ 0 & -1 & 0 \\ +1 & 0 & +1 \end{pmatrix},$$

$$\beta=\frac{1}{\sqrt{3}}, X=-\frac{1}{3} : Q(P)=\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Finally, the two sextets $\Delta_{L,R}$ receive the same treatment as the sextet of model 331 and the electric charges of their components are the same as equations (2.27) and (2.28) but duplicated for left and right. Then

$$Q(\Delta_{L,R})=\begin{pmatrix} 1+\frac{\beta}{\sqrt{3}}+N & \frac{\beta}{\sqrt{3}}+N & \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}\right)+N \\ \frac{\beta}{\sqrt{3}}+N & -1+\frac{\beta}{\sqrt{3}}+N & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}N\right)-N \\ \frac{1}{2}\left(1-\frac{\beta}{\sqrt{3}}\right)+N & -\frac{1}{2}\left(1+\frac{\beta}{\sqrt{3}}\right)-N & -\frac{2\beta}{\sqrt{3}}+N \end{pmatrix}$$

$$=\begin{pmatrix} \frac{2}{3}(1-q)+N & -\frac{1}{3}(1+2q)+N & \frac{1}{3}(2+q)+N \\ -\frac{1}{3}(1+2q)+N & -\frac{2}{3}(2+q)+N & -\frac{1}{3}(1-q)+N \\ \frac{1}{3}(2+q)+N & -\frac{1}{3}(1-q)+N & \frac{2}{3}(1+2q)+N \end{pmatrix} \tag{4.13}$$

For the possible values of β and $\Delta_{L,R} = \eta_{L,R}\eta^T_{L,R}$ $(N(\Delta_{L,R}) = 2N(\eta_{L,R}))$,

$$\begin{aligned}
\beta = -\sqrt{3} \quad \text{or} \quad q = +1, \quad N(\Delta_{L,R}) = 0: \quad & Q(\Delta_{L,R}) = \begin{pmatrix} 0 & -1 & +1 \\ -1 & -2 & 0 \\ +1 & 0 & +2 \end{pmatrix}, \\
\beta = +\sqrt{3} \quad \text{or} \quad q = -2, \quad N(\Delta_{L,R}) = -2: \quad & Q(\Delta_{L,R}) = \begin{pmatrix} 0 & -1 & -2 \\ -1 & -2 & -3 \\ -2 & -3 & -4 \end{pmatrix}, \\
\beta = -\tfrac{1}{\sqrt{3}} \quad \text{or} \quad q = 0, \quad N(\Delta_{L,R}) = -\tfrac{2}{3}: \quad & Q(\Delta_{L,R}) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \\
\beta = \tfrac{1}{\sqrt{3}} \quad \text{or} \quad q = -1, \quad N(\Delta_{L,R}) = -\tfrac{4}{3}: \quad & Q(\Delta_{L,R}) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -2 \end{pmatrix}.
\end{aligned} \tag{4.14}$$

4.4 Vector Bosons

Like in 3221-LR model that has the same vector boson matrix as the Standard Model, only duplicated (one for the right bosons and one for the left ones), the same happens in the 3331-LR model. That is, the $\widetilde{W}_{L,R}$ matrix has the same shape as that of model 331 only in two versions, L and R.

Therefore, the charge operator for 331 version is duplicated here

$$Q(\widetilde{W}_{\mu L,R}) = \left[T^3_{L,R} + \beta T^8_{L,R}, \widetilde{W}_{\mu L,R}\right] \tag{4.15}$$

And the electric charge content is the same as equation (2.15).

	$\beta = \sqrt{3},\ q = -2$	$\beta = 1/\sqrt{3},\ q = -1$	$\beta = -\sqrt{3},\ q = +1$	$\beta = -1/\sqrt{3},\ q = 0$
$\widetilde{W}_{\mu L,R}$	$\begin{pmatrix} 0 & +1 & +2 \\ -1 & 0 & +1 \\ -2 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & +1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & -2 \\ +1 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$

Table 4.2: Electric charge eigenvalues for gauge bosons

Chapter 5

Conclusions

This work studies a method to obtain the electric charge operator to different multiplets from the SSB applied to generators to gauge symmetry groups. From there, the electric charges of particles in gauge theories are all worked out. The method has been tested in four different gauge symmetries and has been successful in all of them, by verifying its validity in the values of electric charges already known and generating confidence in the results for new charges of exotic particles.

On the other hand, the charge eigenstates obtained give us some properties of scalar multiples (especially when these are matrix representations 2×2, 3×3, etc.) when developing new theories, as mentioned for the sextet of 331 and the Φ and P scalar multiples of the 3331-LR model.

In the case of vector bosons, there is an additional contribution: to demonstrate that their electric charges can be calculated in their fundamental representation or in the adjoint one. However, it must be said that the electric charges for the neutral vector bosons we found are those of the gauge eigenstates and not for the mass eigenstates. Despite this, the gauge bosons that shape the vector bosons in their respective mixing might be considered with zero eigenvalues due to the fact that they do not transform in this symmetry.

There are more motivations to continue developing our research in this way, for example we have found relations between hypercharges in 331 models, which makes us think that probably there exists a minimal extension that establish all the charges (including hypercharges) from the beginning.

What follows is to use the method in GUT symmetries proposing new multiplets in an easy and safe way with the new couplings that this entails. Then Lagrangian could be constructed in an easier way than the traditional way. In the group of Phenomenology of High Energies in our faculty the method would be of great help since we worked also analyzing properties of extended gauge theories.

Appendices

Appendix A

Invariance of the vacuum expectation value (vev)

In minimal EWSM with just one scalar doublet, the symmetry $SU(2)_L \otimes U(1)_Y$ is broken to $U(1)_Q$ and Higgs mechanism establish the following

$$\Phi = \begin{pmatrix} \phi^+ \\ v + \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ v + \phi_3 + i\phi_4 \end{pmatrix},$$

where the chosen vacuum is $\langle \Phi \rangle_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$, $v \in \mathbb{R}$. This vacuum has to be invariant under $SU(2)_L \otimes U(1)_Y$.

Recall that the doublet transforms as

$$\begin{aligned} \Phi \to \Phi' &= U_L U_Y \Phi = e^{iT_j \alpha_j(x)} e^{i\frac{Y}{2}f(x)} \Phi \\ &\approx \left(\mathbb{1}_2 + iT_j \alpha_j(x) + i\frac{Y}{2} f(x) \mathbb{1}_2 \right) \Phi = \Phi + \delta\Phi \,. \end{aligned}$$

Assuming the invariance of $\langle\Phi\rangle_0$,

$$\begin{aligned}\delta\langle\Phi\rangle_0 &\approx i\left(T_j\alpha_j(x)+\frac{Y}{2}f(x)\mathbb{1}_2\right)\begin{pmatrix}0\\ v\end{pmatrix}\\ &=\frac{iv}{2}\begin{pmatrix}\alpha_1(x)-i\alpha_2(x)\\ -\alpha_3(x)+f(x)Y\end{pmatrix}\to\begin{pmatrix}0\\ 0\end{pmatrix},\end{aligned} \tag{A.1}$$

from which it follows that $\alpha_1(x)=\alpha_2(x)=0$ and $\alpha_3(x)=f(x)Y$.

Then, the transformation remains as

$$\begin{aligned}\delta\Phi &\approx if(x)\left(T_3+\frac{Y}{2}\mathbb{1}_2\right)\Phi\\ &=ieQ\Phi,\end{aligned} \tag{A.2}$$

where in the last equality $f(x)\to e$ taking into account that the electric charge is the conserved charge of the global symmetry $U(1)_Q$ and $Q=T_3+\frac{Y}{2}\mathbb{1}_2$.

Applying the $U(1)_Q$ gauge transformation on the doublet using $Y(\Phi)=1$,

$$\begin{aligned}\delta\Phi &\approx ie\left(T_3+\frac{1}{2}\mathbb{1}_2\right)\begin{pmatrix}\phi_1+i\phi_2\\ v+\phi_3+i\phi_4\end{pmatrix}\\ &\approx i\begin{pmatrix}+e & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1+i\phi_2\\ v+\phi_3+i\phi_4\end{pmatrix},\end{aligned} \tag{A.3}$$

which tells us that $\phi_1+i\phi_2\to\phi^+$ has an electric charge $+e$ and $\phi_3+i\phi_4\to\phi^0$ does not have one. Namely,

$$\delta(\phi_1+i\phi_2)=+ie\,(\phi_1+i\phi_2)\,;\;\delta(\phi_3+i\phi_4)=0 \tag{A.4}$$

This method can be extended to other symmetries. For example, in the minimal electroweak $SU(3)_L\otimes U(1)_N$ with three scalar triplets

$$\eta = \begin{pmatrix} \eta_a \\ \eta_b \\ \eta_c \end{pmatrix} = \begin{pmatrix} v_\eta + \eta_1 + i\eta_2 \\ \eta_3 + i\eta_4 \\ \eta_5 + i\eta_6 \end{pmatrix} .$$

The chosen vacuum is $\langle \eta \rangle_0 = \begin{pmatrix} v_\eta \\ 0 \\ 0 \end{pmatrix}$, $v_\eta \in \mathbb{R}$. This vacuum has to be invariant under $U(1)_Q$.

Recall that the triplet transforms as

$$\begin{aligned} \eta \to \eta' &= U_L U_N \eta = e^{iT_j \alpha_j(x)} e^{iN_\eta f(x)} \eta \\ &\approx (\mathbb{1}_3 + iT_j\alpha_j(x) + iN_\eta f(x)\mathbb{1}_3)\, \eta = \eta + \delta\eta \,. \end{aligned}$$

Invariance of $\langle \eta \rangle_0$:

$$\begin{aligned} \delta \langle \eta \rangle_0 &\approx i\,(T_j\alpha_j(x) + N_\eta f(x)\mathbb{1}_3) \begin{pmatrix} v_\eta \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{iv_\eta}{2} \begin{pmatrix} 2N_\eta f(x) + \alpha_3 + \frac{\alpha_8}{\sqrt{3}} \\ \alpha_1 + i\alpha_2 \\ \alpha_4 + i\alpha_5 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \end{aligned} \tag{A.5}$$

from which it follows that $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$ and $2N_\eta f(x) + \alpha_3 + \frac{\alpha_8}{\sqrt{3}} = 0$.

The second triplet is

$$\rho = \begin{pmatrix} \rho_a \\ \rho_b \\ \rho_c \end{pmatrix} = \begin{pmatrix} \rho_1 + i\rho_2 \\ v_\rho + \rho_3 + i\rho_4 \\ \rho_5 + i\rho_6 \end{pmatrix} .$$

The chosen vacuum is $\langle \rho \rangle_0 = \begin{pmatrix} 0 \\ v_\rho \\ 0 \end{pmatrix}$, $v_\rho \in \mathbb{R}$. This vacuum has to be invariant under $U(1)_Q$.

The triplet transforms as

$$\rho \to \rho' = U_L U_N \rho \approx (\mathbb{1}_3 + iT_j\alpha_j(x) + iN_\rho f(x)\mathbb{1}_3)\, \rho = \rho + \delta\rho \,.$$

Invariance of $\langle\rho\rangle_0$:

$$\begin{aligned}\delta\langle\rho\rangle_0 &\approx i\left(T_j\alpha_j(x)+N_\eta f(x)\mathbb{1}_3\right)\begin{pmatrix}0\\ v_\rho\\ 0\end{pmatrix}\\ &=\frac{iv_\rho}{2}\begin{pmatrix}\alpha_1-i\alpha_2\\ 2N_\rho f(x)-\alpha_3+\frac{\alpha_8}{\sqrt{3}}\\ \alpha_6+i\alpha_7\end{pmatrix}\to\begin{pmatrix}0\\0\\0\end{pmatrix},\end{aligned}\tag{A.6}$$

from which it follows that $\alpha_1=\alpha_2=\alpha_6=\alpha_7=0$ and $2N_\rho f(x)-\alpha_3+\frac{\alpha_8}{\sqrt{3}}=0$.

And the third triplet,

$$\chi=\begin{pmatrix}\chi_a\\ \chi_b\\ \chi_c\end{pmatrix}=\begin{pmatrix}\chi_1+i\chi_2\\ \chi_3+i\chi_4\\ v_\chi+\chi_5+i\chi_6\end{pmatrix}.$$

The chosen vacuum is $\langle\chi\rangle_0=\begin{pmatrix}0\\0\\v_\chi\end{pmatrix}$, $v_\chi\in\mathbb{R}$. This vacuum has to be invariant under $U(1)_Q$.

As before

$$\chi\to\chi'\approx\left(\mathbb{1}_3+iT_j\alpha_j(x)+iN_\chi f(x)\mathbb{1}_3\right)\chi=\chi+\delta\chi\,.$$

Invariance of $\langle\chi\rangle_0$:

$$\begin{aligned}\delta\langle\chi\rangle_0 &\approx i\left(T_j\alpha_j(x)+N_\chi f(x)\mathbb{1}_3\right)\begin{pmatrix}0\\ 0\\ v_\chi\end{pmatrix}\\ &=\frac{iv_\chi}{2}\begin{pmatrix}\alpha_4-i\alpha_5\\ \alpha_6-i\alpha_7\\ 2N_\chi f(x)-\frac{2\alpha_8}{\sqrt{3}}\end{pmatrix}\to\begin{pmatrix}0\\0\\0\end{pmatrix},\end{aligned}\tag{A.7}$$

from which it follows that $\alpha_4=\alpha_5=\alpha_6=\alpha_7=0$ and $N_\chi f(x)-\frac{\alpha_8}{\sqrt{3}}=0$.

If we put together the results of (A.5), (A.6) and (A.7), we have

$$\begin{aligned}\alpha_1=\alpha_2=\alpha_4=\alpha_5=\alpha_6=\alpha_7=0\\ \text{and}\qquad N_\eta+N_\rho+N_\chi=0.\end{aligned}\tag{A.8}$$

Let us notice that the electric charge content of the triplets depends upon of the values of α_3 and α_8.

Appendix B

Adjoint representations in SU(2) and SU(3) symmetry

In Standard Model, the triplets are transformed with the adjoint representation of $SU(2)$ and are represented as 2×2 matrices which is the appropriate dimensions to be coupled with doublets. Here, the equation (1.3) is not enough for giving electric charges because of the operator dimensions involved. Then, a new transformation has to be performed.

Let a triplet $\mathcal{T} = (t_1, t_2, t_3)^T$ be transformed as $\mathcal{T} \to \mathcal{T}' = \exp[i\theta_j \mathbb{T}_j]\mathcal{T}$, where $\mathbb{T}_j$ is the adjoint representation for $SU(2)$ and spin 1/2. We can demonstrate that this transformation is equivalent to the representation $\widetilde{\mathcal{T}} = \frac{\sigma_j}{2} t_j$ being transformed as $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}' = e^{i\theta_j \frac{\sigma_j}{2}} \widetilde{\mathcal{T}} e^{-i\theta_j \frac{\sigma_j}{2}}$.

That is,

$$
\begin{aligned}
\widetilde{\mathcal{T}}' &= e^{i\theta_j \frac{\sigma_j}{2}} \widetilde{\mathcal{T}} e^{-i\theta_j \frac{\sigma_j}{2}} \\
&\approx \left(\mathbb{1}_2 + i\theta_j \frac{\sigma_j}{2}\right) \widetilde{\mathcal{T}} \left(\mathbb{1}_2 - i\theta_j \frac{\sigma_j}{2}\right) \approx \widetilde{\mathcal{T}} + i\left[\theta_j \frac{\sigma_j}{2}, \widetilde{\mathcal{T}}\right] \\
&= \widetilde{\mathcal{T}} + \frac{1}{2} \begin{pmatrix} t_1\theta_2 - t_2\theta_1 & t_2\theta_3 - t_3\theta_2 - i(t_3\theta_1 - t_1\theta_3) \\ t_2\theta_3 - t_3\theta_2 + i(t_3\theta_1 - t_1\theta_3) & -(t_1\theta_2 - t_2\theta_1) \end{pmatrix}.
\end{aligned}
$$

As $\widetilde{\mathcal{T}} = \frac{1}{2}\begin{pmatrix} t_3 & t_1 - it_2 \\ t_1 + it_2 & -t_3 \end{pmatrix}$, then $\delta\widetilde{\mathcal{T}} \approx \frac{1}{2}\begin{pmatrix} \delta t_3 & \delta t_1 - i\,\delta t_2 \\ \delta t_1 + i\,\delta t_2 & -\,\delta t_3 \end{pmatrix}$. So at first order,

$$\begin{aligned}
\delta t_3 &\approx t_1\theta_2 - t_2\theta_1 \\
\delta t_1 - i\,\delta t_2 &\approx t_2\theta_3 - t_3\theta_2 - i(t_3\theta_1 - t_1\theta_3) \\
\delta t_1 + i\,\delta t_2 &\approx t_2\theta_3 - t_3\theta_2 + i(t_3\theta_1 - t_1\theta_3) \\
-\,\delta t_3 &\approx\ -(t_1\theta_2 - t_2\theta_1).
\end{aligned}$$

Solving for the variations,

$$\begin{pmatrix} \delta t_1 \\ \delta t_2 \\ \delta t_3 \end{pmatrix} \approx \begin{pmatrix} t_2\theta_3 - t_3\theta_2 \\ t_3\theta_1 - t_1\theta_3 \\ t_1\theta_2 - t_2\theta_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}}_{i\theta_j \mathbb{T}_j.} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}, \tag{B.1}$$

Or, $\delta\mathcal{T} \approx i(\theta_j \mathbb{T}_j)\mathcal{T}$, which means that $\mathcal{T}' = \exp[i\theta_j \mathbb{T}_j]\mathcal{T}$.

We conclude that transforming $\mathcal{T}$ with the adjoint representation of $SU(2)$, the same result is obtained if $\widetilde{\mathcal{T}}$ is transformed with the fundamental representation of the same Lie group.

For the $SU(3)$ case, we can define an octet $\mathcal{O} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)^T$ which transforms as $\mathcal{O} \to \mathcal{O}' = \exp\left[i\theta_j \mathbb{T}'_j\right]\mathcal{O}$, where $\mathbb{T}'_j$ are the adjoint representation for $SU(3)$. As before, we can demonstrate that the $\mathcal{O}$ transformation is equivalent to the transformation of $\widetilde{\mathcal{O}} = \frac{1}{2}\lambda_j t_j$. That is $\widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}' = e^{i\theta_j \frac{\lambda_j}{2}} \widetilde{\mathcal{O}} e^{-i\theta_j \frac{\lambda_j}{2}}$.

$$\begin{aligned}
\widetilde{\mathcal{O}}' &= e^{i\theta_j\lambda_j} \widetilde{\mathcal{O}} e^{-i\theta_j\lambda_j} \\
&\approx \left(\mathbb{1}_3 + i\theta_j \frac{\lambda_j}{2}\right) \widetilde{\mathcal{O}} \left(\mathbb{1}_3 - i\theta_j \frac{\lambda_j}{2}\right) \approx \widetilde{\mathcal{O}} + i\left[\theta_j \frac{\lambda_j}{2}, \widetilde{\mathcal{O}}\right] \\
&= \widetilde{\mathcal{O}} + \frac{i}{2}\mathcal{M},
\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{M}_{11} &= \theta_1(-t_2)+\theta_2 t_1-\theta_4 t_5+\theta_5 t_4\\
\mathcal{M}_{12} &= -i\theta_1 t_3-\theta_2 t_3+i\theta_3 t_1+\theta_3 t_2+\frac{1}{2}i\theta_4 t_6+\frac{\theta_5 t_6}{2}+\frac{1}{2}i\theta_5 t_7-\frac{1}{2}i\theta_6 t_4+\frac{\theta_7 t_4}{2}-\frac{1}{2}i\theta_7 t_5-\frac{\theta_4 t_7}{2}-\frac{\theta_6 t_5}{2}\\
\mathcal{M}_{13} &= \frac{1}{2}i\theta_1 t_6+\frac{\theta_1 t_7}{2}+\frac{\theta_2 t_6}{2}-\frac{1}{2}i\theta_2 t_7+\frac{1}{2}i\theta_3 t_4+\frac{\theta_3 t_5}{2}-\frac{1}{2}i\theta_4 t_3-\frac{1}{2}i\sqrt{3}\theta_4 t_8-\frac{1}{2}\sqrt{3}\theta_5 t_8-\frac{1}{2}i\theta_6 t_1\\
&\quad+\frac{1}{2}i\theta_7 t_2+\frac{1}{2}i\sqrt{3}\theta_8 t_4+\frac{1}{2}\sqrt{3}\theta_8 t_5-\frac{\theta_5 t_3}{2}-\frac{\theta_6 t_2}{2}-\frac{\theta_7 t_1}{2}\\
\mathcal{M}_{21} &= i\theta_1 t_3-\theta_2 t_3-i\theta_3 t_1+\theta_3 t_2-\frac{1}{2}i\theta_4 t_6+\frac{\theta_5 t_6}{2}-\frac{1}{2}i\theta_5 t_7+\frac{1}{2}i\theta_6 t_4+\frac{\theta_7 t_4}{2}+\frac{1}{2}i\theta_7 t_5-\frac{\theta_4 t_7}{2}-\frac{\theta_6 t_5}{2}\\
\mathcal{M}_{22} &= \theta_1 t_2-\theta_2 t_1-\theta_6 t_7+\theta_7 t_6\\
\mathcal{M}_{23} &= \frac{1}{2}i\theta_1 t_4+\frac{\theta_1 t_5}{2}+\frac{1}{2}i\theta_2 t_5-\frac{1}{2}i\theta_3 t_6-\frac{1}{2}i\theta_4 t_1+\frac{\theta_4 t_2}{2}-\frac{1}{2}i\theta_5 t_2+\frac{1}{2}i\theta_6 t_3-\frac{1}{2}i\sqrt{3}\theta_6 t_8+\frac{\theta_7 t_3}{2}\\
&\quad-\frac{1}{2}\sqrt{3}\theta_7 t_8+\frac{1}{2}i\sqrt{3}\theta_8 t_6+\frac{1}{2}\sqrt{3}\theta_8 t_7-\frac{\theta_2 t_4}{2}-\frac{\theta_3 t_7}{2}-\frac{\theta_5 t_1}{2}\\
\mathcal{M}_{31} &= -\frac{1}{2}i\theta_1 t_6+\frac{\theta_1 t_7}{2}+\frac{\theta_2 t_6}{2}+\frac{1}{2}i\theta_2 t_7-\frac{1}{2}i\theta_3 t_4+\frac{\theta_3 t_5}{2}+\frac{1}{2}i\theta_4 t_3+\frac{1}{2}i\sqrt{3}\theta_4 t_8-\frac{1}{2}\sqrt{3}\theta_5 t_8+\frac{1}{2}i\theta_6 t_1\\
&\quad-\frac{1}{2}i\theta_7 t_2-\frac{1}{2}i\sqrt{3}\theta_8 t_4+\frac{1}{2}\sqrt{3}\theta_8 t_5-\frac{\theta_5 t_3}{2}-\frac{\theta_6 t_2}{2}-\frac{\theta_7 t_1}{2}\\
\mathcal{M}_{32} &= -\frac{1}{2}i\theta_1 t_4+\frac{\theta_1 t_5}{2}-\frac{1}{2}i\theta_2 t_5+\frac{1}{2}i\theta_3 t_6+\frac{1}{2}i\theta_4 t_1+\frac{\theta_4 t_2}{2}+\frac{1}{2}i\theta_5 t_2-\frac{1}{2}i\theta_6 t_3+\frac{1}{2}i\sqrt{3}\theta_6 t_8+\frac{\theta_7 t_3}{2}\\
&\quad-\frac{1}{2}\sqrt{3}\theta_7 t_8-\frac{1}{2}i\sqrt{3}\theta_8 t_6+\frac{1}{2}\sqrt{3}\theta_8 t_7-\frac{\theta_2 t_4}{2}-\frac{\theta_3 t_7}{2}-\frac{\theta_5 t_1}{2}\\
\mathcal{M}_{33} &= \theta_4 t_5-\theta_5 t_4+\theta_6 t_7-\theta_7 t_6
\end{aligned}
\tag{B.2}
$$

Since $\widetilde{\mathcal{O}}=\frac{1}{2}\lambda_j t_j=\frac{1}{2}\begin{pmatrix} t_3+\frac{t_8}{\sqrt{3}} & t_1-it_2 & t_4-it_5\\ t_1+it_2 & \frac{t_8}{\sqrt{3}}-t_3 & t_6-it_7\\ t_4+it_5 & t_6+it_7 & -\frac{2t_8}{\sqrt{3}}\end{pmatrix}$, then

$$
\begin{aligned}
\delta t_3+\tfrac{\delta t_8}{\sqrt{3}}&=2\mathcal{M}_{11}, &\quad \delta t_1-i\delta t_2&=2\mathcal{M}_{12}, &\quad \delta t_4-i\delta t_5&=2\mathcal{M}_{13},\\
\delta t_1+i\delta t_2&=2\mathcal{M}_{21}, & -\delta t_3+\tfrac{\delta t_8}{\sqrt{3}}&=2\mathcal{M}_{22}, & \delta t_6-i\delta t_7&=2\mathcal{M}_{23},\\
\delta t_4+i\delta t_5&=2\mathcal{M}_{31}, & \delta t_6+i\delta t_7&=2\mathcal{M}_{32}, & -\tfrac{2\delta t_8}{\sqrt{3}}&=2\mathcal{M}_{33}.
\end{aligned}
$$

Solving for the variations,

$$
\begin{pmatrix} \delta t_1 \\ \delta t_2 \\ \delta t_3 \\ \delta t_4 \\ \delta t_5 \\ \delta t_6 \\ \delta t_7 \\ \delta t_8 \end{pmatrix} \approx \begin{pmatrix}
t_2\theta_3 - t_3\theta_2 + \frac{1}{2}(t_6\theta_5 - t_5\theta_6 + t_4\theta_7 - t_7\theta_4) \\
t_3\theta_1 - t_1\theta_3 + \frac{1}{2}(t_4\theta_6 - t_6\theta_4 + t_5\theta_7 - t_7\theta_5) \\
t_1\theta_2 - t_2\theta_1 + \frac{1}{2}(t_4\theta_5 - t_5\theta_4 + t_7\theta_6 - t_6\theta_7) \\
\frac{1}{2}(t_1\theta_6 - t_6\theta_1 + t_7\theta_2 - t_2\theta_7 + t_3\theta_4 - t_4\theta_3 + \sqrt{3}(t_5\theta_8 - t_8\theta_5)) \\
\frac{1}{2}(t_3\theta_4 - t_4\theta_3 + t_1\theta_6 - t_6\theta_1 + t_7\theta_2 - t_2\theta_7 + \sqrt{3}(t_8\theta_4 - t_4\theta_8)) \\
\frac{1}{2}(t_1\theta_5 - t_5\theta_1 + t_2\theta_4 - t_4\theta_2 + t_3\theta_7 - t_7\theta_3 + \sqrt{3}(t_7\theta_8 - t_8\theta_7)) \\
\frac{1}{2}(t_1\theta_4 - t_4\theta_1 + t_2\theta_5 - t_5\theta_2 + t_3\theta_6 - t_6\theta_3 + \sqrt{3}(t_8\theta_6 - t_6\theta_8)) \\
\frac{\sqrt{3}}{2}(t_4\theta_5 - t_5\theta_4 + t_6\theta_7 - t_7\theta_6)
\end{pmatrix}
$$

$$
= \underbrace{\begin{pmatrix}
0 & \theta_3 & -\theta_2 & \frac{\theta_7}{2} & -\frac{\theta_6}{2} & \frac{\theta_5}{2} & -\frac{\theta_4}{2} & 0 \\
-\theta_3 & 0 & \theta_1 & \frac{\theta_6}{2} & \frac{\theta_7}{2} & -\frac{\theta_4}{2} & -\frac{\theta_5}{2} & 0 \\
\theta_2 & -\theta_1 & 0 & \frac{\theta_5}{2} & -\frac{\theta_4}{2} & -\frac{\theta_7}{2} & \frac{\theta_6}{2} & 0 \\
-\frac{\theta_7}{2} & -\frac{\theta_6}{2} & -\frac{\theta_5}{2} & 0 & \frac{\theta_3}{2} + \frac{\sqrt{3}\theta_8}{2} & \frac{\theta_2}{2} & \frac{\theta_1}{2} & -\frac{1}{2}\sqrt{3}\theta_5 \\
\frac{\theta_6}{2} & -\frac{\theta_7}{2} & \frac{\theta_4}{2} & -\frac{\theta_3}{2} - \frac{\sqrt{3}\theta_8}{2} & 0 & -\frac{\theta_1}{2} & \frac{\theta_2}{2} & \frac{\sqrt{3}\theta_4}{2} \\
-\frac{\theta_5}{2} & \frac{\theta_4}{2} & \frac{\theta_7}{2} & -\frac{\theta_2}{2} & \frac{\theta_1}{2} & 0 & \frac{\sqrt{3}\theta_8}{2} - \frac{\theta_3}{2} & -\frac{1}{2}\sqrt{3}\theta_7 \\
\frac{\theta_4}{2} & \frac{\theta_5}{2} & -\frac{\theta_6}{2} & -\frac{\theta_1}{2} & -\frac{\theta_2}{2} & \frac{\theta_3}{2} - \frac{\sqrt{3}\theta_8}{2} & 0 & \frac{\sqrt{3}\theta_6}{2} \\
0 & 0 & 0 & \frac{\sqrt{3}\theta_5}{2} & -\frac{1}{2}\sqrt{3}\theta_4 & \frac{\sqrt{3}\theta_7}{2} & -\frac{1}{2}\sqrt{3}\theta_6 & 0
\end{pmatrix}}_{i\theta_j \mathbb{T}'_j} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \end{pmatrix}. \tag{B.3}
$$

It means that the transformation in the adjoint representation $\delta\mathcal{O} \approx i\theta_j \mathbb{T}'_j \mathcal{O}$ or $\mathcal{O}' = \exp\left[i\theta_j \mathbb{T}'_j\right]\mathcal{O}$, gets the same result as the transformation $\widetilde{\mathcal{O}}' = e^{i\theta_j \frac{\lambda_j}{2}} \widetilde{\mathcal{O}} e^{-i\theta_j \frac{\lambda_j}{2}}$.

Appendix C

Charge Operator

C.1 Conserved charge in $\mathrm{U(1)_Q}$

An example of a gauge theory with $U(1)$ symmetry is QED, whose lagrangian is

$$\mathcal{L} = \overline{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{C.1}$$

In global case, $D_\mu \to \partial_\mu$ and fermionic fields transform like $\psi'(x) = e^{iqf}\psi(x)$, with constants q and f. Then $\delta\psi(x) \approx iqe\psi(x)$, where $f \to e$. Besides, in this case $\delta A_\mu(x) = 0$ and tensor $F^{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ does not transform.

According to Noether's theorem, the conserved current in this symmetry is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\psi = -eq\overline{\psi}\gamma^\mu\psi. \tag{C.2}$$

The gauge group has a generator Q with eigenvalues q. In quantized theory, the operator $\widehat{Q}$is

$$\widehat{Q} = \int d^3x J^0 = -eq\int d^3x\psi^\dagger\psi, \tag{C.3}$$

which is called charge operator, and if it is written in terms of raising and lowering operators, results the net charge of particles and antiparticles, according with equation (C.8).

On the other hand, in the local case the fields are transformed $\phi'(x) = e^{iqf(x)}\phi(x)$, with

$\delta\phi(x) = iqf(x)\phi(x)$. In this case, the associated gauge field is transformed $A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu f(x)$.

Here, the conserved current with respect to the part of the Lagrangian that corresponds to Maxwell $F_{\mu\nu}F^{\mu\nu}$ is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu = -\frac{1}{2}F^{\mu\nu}\partial_\nu f(x). \tag{C.4}$$

That is, Noether's current can be assumed as $F^{\mu\nu}\partial_\nu f(x)$. It is known that for gauge symmetries (local) the second Noether theorem is applied where conserved charges are not obtained; these latter only come from global symmetries applying Noether's first theorem. However, it can be shown that for the QED there is a relationship between the local symmetry $U(1)$ and the electric charge.

$$\begin{aligned} Q &= \int J^0 d^3x = \int F^{0i}\partial_i f(x) d^3x \\ &= \int \partial_i \left(F^{0i} f(x)\right) d^3x - \int (\partial_i F^{0i}) f(x) d^3x \,. \end{aligned}$$

But $\partial_\nu F^{\mu\nu} = 0$, then $\partial_i F^{\mu i} = -\partial_0 F^{\mu 0}$. For $\mu = 0$, $\partial_i F^{0i} = -\partial_0 F^{00} = 0$. That means that the last integral has no contribution and the charge left

$$Q = \int \partial_i \left(f(x)F^{0i}\right) d^3x = -\oint_S f(x)\vec{E}\cdot\vec{n}\, dS \,, \tag{C.5}$$

because $F^{0i} = -E_i$. If $S \to \infty$, the principle of charge conservation requires that $f(x)$ be a constant in S, so that Q is proportional to the electric field flow and Gauss's law is applied,

$$Q = -e\oint_S \vec{E}\cdot\vec{n}\, dS \; = -eq, \tag{C.6}$$

making $f \to e$. This result is identical to that of (C.3) with adequate normalization of $\int d^3x\psi^\dagger\psi$.

C.2 Particle-Antiparticle

The total charge of a spinor is $Q = q\int d^3x\psi^\dagger\psi$, with $\left\{\psi^\dagger(x), \psi(y)\right\} = \delta^3(x-y)$ y $\{\psi(x), \psi(y)\} = 0$. Then,

$$\begin{aligned}
\psi(x)Q &= q\int d^3y\psi(x)\psi^\dagger(y)\psi(y) \\
&= q\int d^3y\delta^3(x-y)\psi(y) - q\int d^3y\psi^\dagger(y)\psi(x)\psi(y) \\
&= q\psi(x) + Q\psi(x) \\
Q\psi^\dagger(x) &= q\int d^3y\psi^\dagger(y)\psi(y)\psi^\dagger(x) \\
&= q\int d^3y\psi^\dagger(y)\delta^3(x-y) + q\int d^3y\psi^\dagger(x)\psi^\dagger(y)\psi(y) \\
&= q\psi^\dagger(x) + \psi^\dagger(x)Q
\end{aligned}$$

Therefore,

$$[Q, \psi] = -q\psi, \qquad \left[Q, \psi^\dagger\right] = q\psi^\dagger. \tag{C.7}$$

If it is assumed that $|\psi\rangle = \psi^\dagger|0\rangle$ and $\left|\overline{\psi}\right\rangle = \psi|0\rangle$,

$$\begin{aligned}
Q|\psi\rangle = Q\psi^\dagger|0\rangle = \left(q\psi^\dagger + \psi^\dagger Q\right)|0\rangle = q\psi^\dagger|0\rangle = q|\psi\rangle \\
Q\left|\overline{\psi}\right\rangle = Q\psi|0\rangle = (-q\psi + \psi Q)|0\rangle = -q\psi|0\rangle = -q|\psi\rangle
\end{aligned}$$

It can be seen that Q has eigenvalues of opposite sign for particles and antiparticles. So, Q applied to a particle system measures the total charge

$$Q = q\left(N_\psi - N_{\overline{\psi}}\right). \tag{C.8}$$

Appendix D

Flavor Symmetries

D.1 Isospin Symmetry $\mathrm{SU}(2)$

Historically, Heisenberg (1932) [50] implemented the idea of considering the proton and neutron as states of the same particle, the nucleon.

In this way, $p = \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ and $n = \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$, or $p = (1\ \ 0)^T$ and $n = (0\ \ 1)^T$ in an isospin space with the usual $SU(2)$ transformations in which strong force is conserved. If we put them together to form a composite particle, we have to add their spins.

In the same foot, if we consider another isodoublet $(u\ \ d)^T$ or anti-isodoublet $(-\bar{d}\ \ \bar{u})$[1], with $u = \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$, $d = \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$, $\bar{u} = \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$ and $\bar{d} = -\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$. From the Clebsch-Gordan table, we get composite representations constituted by an isospin-1 triplet (symmetric) and an isospin-0 singlet (antisymmetric).

$$
\left.\begin{aligned}
\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= |1\ 1\rangle \\
\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}|1\ 0\rangle + \tfrac{1}{\sqrt{2}}|0\ 0\rangle \\
\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}|1\ 0\rangle - \tfrac{1}{\sqrt{2}}|0\ 0\rangle \\
\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= |1\ -1\rangle
\end{aligned}\right\}
\begin{aligned}
|1\ 1\rangle &= \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \\
|1\ 0\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right) \\
|1\ -1\rangle &= \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \\
\hline
|0\ 0\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)
\end{aligned}
\tag{D.1}
$$

[1]See Halzen, F. and Martin, A. D. **(1984)** *Quarks and Leptons*, p.42

Evidently, the triplet and the singlet shown are gotten from $\mathbf{2}\otimes\mathbf{2}=\mathbf{3}\oplus\mathbf{1}$. In the "up-down" representation, we obtain the isotriplet of pions $(\pi^+ \ \ \pi^0 \ \ \pi^-)^T$ and the isosinglet ω,

$$\begin{aligned} |1\ 1\rangle &= \pi^+ = -\,u\bar{d} \\ |1\ 0\rangle &= \pi^{0[2]} = (u\bar{u}-d\bar{d})/\sqrt{2} \\ |1\ -1\rangle &= \pi^- = d\bar{u} \\ |0\ 0\rangle &= \omega = (u\bar{u}+d\bar{d})/\sqrt{2} \end{aligned} \tag{D.2}$$

If we want to add one more isodoublet ro form some baryon quartet, we may perform $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}=(\mathbf{3}\oplus\mathbf{1})\otimes\mathbf{2}=\mathbf{4}\oplus\mathbf{2}\oplus\mathbf{2}$. Again, from C-G table:

$$\begin{aligned}
|1\ 1\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle \\
|1\ 1\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle+\sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \\
|1\ 0\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{3}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \\
|1\ 0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle+\tfrac{1}{\sqrt{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \\
|1\ -1\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle-\sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \\
|1\ -1\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle \\
|0\ 0\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \\
|0\ 0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle
\end{aligned}$$

[2]This combination also get ρ^0.

Solving, we have one quartet and two doublets:

$$\begin{aligned}
\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle &= \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\\
\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)\\
\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)\\
\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle &= \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\\
\hline
\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{6}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)\\
\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{6}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\right)-\sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\\
\hline
\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle-\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)\\
\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle-\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\right)
\end{aligned} \tag{D.3}$$

In the "up-down" representation for the quartet, we obtain the Δ-baryon quartet

$$\begin{aligned}
\left|\frac{3}{2}\ \frac{3}{2}\right\rangle &= \Delta^{++} = uuu\\
\left|\frac{3}{2}\ \frac{1}{2}\right\rangle &= \Delta^{+} = \frac{1}{\sqrt{3}}(uud+udu+duu)\\
\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle &= \Delta^{0} = \frac{1}{\sqrt{3}}(udd+dud+ddu)\\
\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle &= \Delta^{-} = ddd
\end{aligned} \tag{D.4}$$

Appendix E

Electric charge and gauge coupling constants

E.1 Standard Model 321 symmetry

In SM, gauge coupling constants are related with electric charge since the SSM $SU(2)_L \otimes U(1)_Y$ turn into $U(1)$ including their respective coupling constants.

One way to achieve this is comparing the coefficients of known interactions with their QED counterparts where the electric charge appears everywhere.

When the derivatives of Higgs sector are developed with $Y(\Phi) = +1$ and $\langle\Phi\rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$ taking into account the neutral field interactions only,

$$
\begin{aligned}
\left[D^\mu \Phi\right]^\dagger \left[D_\mu \Phi\right] &= \left[\left(\partial^\mu + ig_L W^{\mu j}\frac{\sigma_j}{2} + ig_Y \frac{Y}{2} B^\mu\right)\Phi\right]^\dagger \left[\left(\partial_\mu + ig_L W_{\mu j}\frac{\sigma_j}{2} + ig_Y \frac{Y}{2} B_\mu\right)\Phi\right] \\
&= \ldots \frac{g_L^2}{4}\left(W_j^\mu \frac{\sigma_j}{2}\Phi\right)^\dagger \left(W_{\mu j}\frac{\sigma_j}{2}\Phi\right) + \frac{g_L g_Y}{4}\left[\left(W_j^\mu \frac{\sigma_j}{2}\right)^\dagger (B_\mu \Phi) + (B^\mu \Phi)^\dagger \left(W_j^\mu \frac{\sigma_j}{2}\right)\right] \\
&\quad + \frac{g_Y^2}{4}(B^\mu \Phi)^\dagger (B_\mu \Phi) \\
&= \ldots \frac{g_L^2 v^2}{8}\left(W^{\mu +}W_\mu^- + W_3^\mu W_{3\mu}\right) - \frac{g_L g_Y v^2}{8} B^\mu W_{3\mu} - \frac{g_L g_Y v^2}{8} W_3^\mu B_\mu + \frac{g_Y^2 v^2}{8} B^\mu B_\mu \\
&= \ldots \frac{g_L^2 v^2}{8} W^{\mu +}W_\mu^- + \frac{v^2}{8}\begin{pmatrix} W_3^\mu & B^\mu \end{pmatrix}\begin{pmatrix} g_L^2 & -g_L g_Y \\ -g_L g_Y & g_Y^2 \end{pmatrix}\begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}.
\end{aligned}
\tag{E.1}
$$

On the penultimate line the vev of Φ was applied and in the last line the central matrix must be diagonal in order to get the mass terms for W_3^μ and B^μ. This means that these gauge bosons are not physical ones.

The mass for charged W boson is $M_W = \dfrac{g_L v}{2}$ and for the neutral ones, the symmetric matrix can be diagonalized by an orthogonal matrix $\begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}$, where θ_W is called weak mixing or Weinberg angle. The resulting mass eigenvectors are

$$\begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix} \rightarrow \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} W_{3\mu}\cos\theta_W - B_\mu \sin\theta_W \\ W_{3\mu}\sin\theta_W + B_\mu\cos\theta_W \end{pmatrix}, \tag{E.2}$$

with $\tan\theta_W = \frac{g_Y}{g_L}$. The corresponding mass eigenvalues are

$$M_Z = \frac{v}{2}\sqrt{g_L^2 + g_Y^2},\; M_A = 0 \tag{E.3}$$

For the charged gauge bosons, let us take into account only the W_μ^+ and W_μ^- interactions in (E.1),

$$[D^\mu\Phi]^\dagger [D_\mu\Phi] = \ldots \frac{g_L^2}{4}\left(W_j^\mu \frac{\sigma_j}{2}\Phi\right)^\dagger \left(W^{\mu j}\frac{\sigma_j}{2}\Phi\right) = \ldots \frac{g_L^2 v^2}{8} W^{\mu +} W_\mu^- . \tag{E.4}$$

In this case, the mass term has been obtained directly and $M_W = \frac{g_L v}{2}$. Then the W/Z mass ratio is

$$\frac{M_W^2}{M_Z^2} = \frac{g_L^2}{g_L^2 + g_Y^2} = \cos^2\theta_W . \tag{E.5}$$

In order to get the electric charge, we may find the QED interaction term between the left electromagnetic current for electrons and photon $e(\bar{e}_L\gamma^\mu e_L)A_\mu$. The left-handed doublet has $Y(L) = -1$ and the right-handed singlet $Y(R) = -2$ for $R = e$ in the leptonic sector,

$$\begin{aligned} i\left[\bar{L}_l \not{D} L_l + \bar{R}_l \not{D} R_l\right] &= i\left[\bar{L}\gamma^\mu\left(\partial_\mu + i g_L W_{\mu j}\frac{\sigma_j}{2} - i\frac{g_Y}{2}B_\mu\right)L + \bar{e}_R\gamma^\mu\left(\partial_\mu + i g_L W_{\mu j}\frac{\sigma_j}{2} - i g_Y B_\mu\right)e_R\right] \\ &= \ldots \frac{g_L}{2}\bar{e}_L\gamma^\mu e_L W_{3\mu} + \frac{g_Y}{2}\bar{e}_L\gamma^\mu e_L B_\mu + \frac{g_Y}{2}\bar{e}_R\gamma^\mu e_R B_\mu \\ &= \ldots \frac{g_L}{2}\bar{e}_L\gamma^\mu e_L\left(\cos\theta_W Z_\mu + \sin\theta_W A_\mu\right) + \frac{g_Y}{2}\bar{e}_L\gamma^\mu e_L\left(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu\right) \\ &= \ldots \left(\frac{g_L}{2}\sin\theta_W + \frac{g_Y}{2}\cos\theta\right)\bar{e}_L\gamma^\mu e_L A_\mu . \end{aligned} \tag{E.6}$$

The coefficient in the last line is the electric charge $e = \frac{g_L}{2}\sin\theta_W + \frac{g_Y}{2}\cos\theta_W$. Since $\tan\theta_W = \frac{g_Y}{g_L}$, then

$$e = g_Y\cos\theta_W = g_L\sin\theta_W . \tag{E.7}$$

From the last relation,

$$e = \frac{g_Y g_L}{\sqrt{g_Y^2 + g_L^2}} \rightarrow \boxed{\frac{1}{e^2} = \frac{1}{g_L^2} + \frac{1}{g_Y^2}} \tag{E.8}$$

E.2 Extension: 331 symmetry

As in the SM we can write the covariant derivative for triplets,

$$D_\mu \Phi_i = \partial_\mu \Phi_i + \frac{ig_{3L}}{2} \begin{pmatrix} W_{3\mu} + \frac{1}{\sqrt{3}} W_{8\mu} & W_{1\mu} - iW_{2\mu} & W_{4\mu} - iW_{5\mu} \\ W_{1\mu} + iW_{2\mu} & -W_{3\mu} + \frac{1}{\sqrt{3}} W_{8\mu} & W_{6\mu} - iW_{7\mu} \\ W_{4\mu} + iW_{5\mu} & W_{6\mu} + iW_{7\mu} & -\frac{2}{\sqrt{3}} W_{8\mu} \end{pmatrix} \Phi_i + ig_N B_\mu N_{\Phi_i} \Phi_i, \tag{E.9}$$

where $\Phi_1 = \eta, \Phi_2 = \rho, \Phi_3 = \chi$; and hypercharges N_{Φ_i} are defined in

The eigenstates for the charged (complex) vector bosons are,

$$W_\mu^+ = \frac{1}{\sqrt{2}} \left(W_{1\mu} - iW_{2\mu} \right), \; V_\mu^+ = \frac{1}{\sqrt{2}} \left(W_{4\mu} + iW_{5\mu} \right), \; U_\mu^{++} = \frac{1}{\sqrt{2}} \left(W_{6\mu} + iW_{7\mu} \right). \tag{E.10}$$

Then, the vector boson matrix is

$$\widetilde{W}_\mu = \begin{pmatrix} \frac{W_{3\mu}}{2} + \frac{W_{8\mu}}{2\sqrt{3}} & \frac{W_\mu^+}{\sqrt{2}} & \frac{V_\mu^-}{\sqrt{2}} \\ \frac{W_\mu^-}{\sqrt{2}} & -\frac{W_{3\mu}}{2} + \frac{W_{8\mu}}{2\sqrt{3}} & \frac{U^{--}}{\sqrt{2}} \\ \frac{V_\mu^+}{\sqrt{2}} & \frac{U^{++}}{\sqrt{2}} & -\frac{W_{8\mu}}{\sqrt{3}} \end{pmatrix}.$$

Let us build up the vector boson mass matrix. For $\eta \sim (\mathbf{3_L}, 0)$ and $\langle \eta \rangle_0 = \begin{pmatrix} v_\eta & 0 & 0 \end{pmatrix}^T$,

$$\begin{aligned} \left[D^\mu \eta \right]^\dagger \left[D_\mu \eta \right] &= \left[\left(\partial^\mu + ig_{3L} W^{\mu j} T_j \right) \eta \right]^\dagger \left[\left(\partial_\mu + ig_{3L} W_{\mu j} T_j \right) \eta \right] \\ &= \ldots g_{3L}^2 \left(W_j^\mu T_j \eta \right)^\dagger \left(W_{\mu j} T_j \eta \right) \\ &= \ldots \frac{g_{3L}^2 v_\eta^2}{4} \left(W_3^\mu W_{3\mu} + W_3^\mu \frac{W_{8\mu}}{\sqrt{3}} + \frac{W_8^\mu}{\sqrt{3}} W_{3\mu} + \frac{W_8^\mu}{\sqrt{3}} \frac{W_{8\mu}}{\sqrt{3}} \right) + \frac{g_{3L}^2 v_\eta^2}{2} \left(W^{\mu +} W_\mu^- + V^{\mu -} V_\mu^+ \right) \\ &= \ldots + \frac{g_{3L}^2 v_\eta^2}{4} \begin{pmatrix} W_3^\mu & W_8^\mu & B^\mu \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_{3\mu} \\ W_{8\mu} \\ B_\mu \end{pmatrix} + \frac{g_{3L}^2 v_\eta^2}{2} \left(W^{\mu +} W_\mu^- + V^{\mu -} V_\mu^+ \right). \end{aligned} \tag{E.11}$$

For $\rho \sim (\mathbf{3_L}, +1)$ and $\langle\rho\rangle_0 = \begin{pmatrix} 0 & v_\rho & 0 \end{pmatrix}^T$,

$$
\begin{aligned}
&[D^\mu\rho]^\dagger [D_\mu\rho] = \left[\left(\partial^\mu + ig_{3L}W^{\mu j}T_j + ig_N B^\mu\right)\rho\right]^\dagger \left[\left(\partial_\mu + ig_{3L}W_{\mu j}T_j + ig_N B_\mu\right)\rho\right] \\
&= \dots g_{3L}^2 \left(W_j^\mu T_j\rho\right)^\dagger \left(W_{\mu j}T_j\rho\right) + g_{3L}g_N \left[\left(W_j^\mu T_j\rho\right)^\dagger (B_\mu\rho) + (B^\mu\rho)^\dagger \left(W_{\mu j}T_j\rho\right)\right] + g_N^2 (B^\mu\rho)^\dagger (B_\mu\rho) \\
&= \dots \frac{g_{3L}^2 v_\rho^2}{4}\left(W_3^\mu W_{3\mu} - W_3^\mu \frac{W_{8\mu}}{\sqrt{3}} - \frac{W_8^\mu}{\sqrt{3}}W_{3\mu} + \frac{W_8^\mu}{\sqrt{3}}\frac{W_{8\mu}}{\sqrt{3}}\right) + \frac{g_{3L}^2 v_\rho^2}{2}\left(W^{\mu+}W_\mu^- + U^{\mu++}U_\mu^{--}\right) \\
&\quad + \frac{g_{3L}g_N v_\rho^2}{2}\left(-W_3^\mu B_\mu + \frac{W_8^\mu}{\sqrt{3}}B_\mu - B^\mu W_{3\mu} + B^\mu \frac{W_{8\mu}}{\sqrt{3}}\right) + \frac{g_N^2}{2}v_\rho^2 (B^\mu B_\mu) \\
&= \dots \frac{g_{3L}^2 v_\rho^2}{4}\begin{pmatrix} W_3^\mu & W_8^\mu & B^\mu \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} & -2t \\ -\frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{2t}{\sqrt{3}} \\ -2t & \frac{2t}{\sqrt{3}} & 4t^2 \end{pmatrix}\begin{pmatrix} W_{3\mu} \\ W_{8\mu} \\ B_\mu \end{pmatrix} + \frac{g_{3L}^2 v_\rho^2}{2}\left(W^{\mu+}W_\mu^- + U^{\mu++}U_\mu^{--}\right),
\end{aligned}
\tag{E.12}
$$

where $t = \frac{g_N}{g_{3L}}$.

For $\chi \sim (\mathbf{3_L}, -1)$ and $\langle\chi\rangle_0 = \begin{pmatrix} v_\chi & 0 & 0 \end{pmatrix}^T$,

$$
\begin{aligned}
&[D^\mu\chi]^\dagger [D_\mu\chi] = \left[\left(\partial^\mu + ig_{3L}W^{\mu j}T_j - ig_N B^\mu\right)\chi\right]^\dagger \left[\left(\partial_\mu + ig_{3L}W_{\mu j}T_j - ig_N B_\mu\right)\chi\right] \\
&= \dots g_{3L}^2 \left(W_j^\mu T_j\chi\right)^\dagger \left(W_{\mu j}T_j\chi\right) - g_{3L}g_N \left[\left(W_j^\mu T_j\chi\right)^\dagger (B_\mu\chi) + (B^\mu\rho)^\dagger \left(W_{\mu j}T_j\chi\right)\right] + g_N^2 (B^\mu\chi)^\dagger (B_\mu\chi) \\
&= \dots g_{3L}^2 v_\chi^2 \left(\frac{W_8^\mu}{\sqrt{3}}\frac{W_{8\mu}}{\sqrt{3}}\right) + \frac{g_{3L}g_N v_\chi^2}{2}\left(\frac{W_8^\mu}{\sqrt{3}}B_\mu + B^\mu\frac{W_{8\mu}}{\sqrt{3}}\right) + \frac{g_N^2 v_\chi^2}{2}(B^\mu B_\mu) \\
&\quad + \frac{g_{3L}^2 v_\chi^2}{2}\left(V^{\mu+}V_\mu^- + U^{\mu++}U_\mu^{--}\right) \\
&= \dots \frac{g_{3L}^2 v_\chi^2}{4}\begin{pmatrix} W_3^\mu & W_8^\mu & B^\mu \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3} & \frac{4t}{\sqrt{3}} \\ 0 & \frac{4t}{\sqrt{3}} & 4t^2 \end{pmatrix}\begin{pmatrix} W_{3\mu} \\ W_{8\mu} \\ B_\mu \end{pmatrix} + \frac{g_{3L}^2 v_\chi^2}{2}\left(V^{\mu+}V_\mu^- + U^{\mu++}U_\mu^{--}\right).
\end{aligned}
\tag{E.13}
$$

For the sextet $S \sim (\mathbf{6_L}, 0)$ and $\langle S \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & 0 & v_2 \\ 0 & v_2 & 2 \end{pmatrix}$,

$$\begin{aligned}
&\mathrm{Tr}\left[(D^\mu S)^\dagger (D_\mu S)\right] = \mathrm{Tr}\left[\partial_\mu S + i g_{3L}\left(W_j^\mu T_j S + S W_j^\mu T_j\right)\right]^\dagger \left[\partial_\mu S + i g_{3L}\left(W_j^\mu T_j S + S W_j^\mu T_j\right)\right] \\
&= \dots g_{3L}^2 \left[\left(W_j^\mu T_j S\right)^\dagger (W_{\mu j} T_j S) + \left(W_j^\mu T_j S\right)^\dagger (S W_{\mu j} T_j) + \left(S W_j^\mu T_j\right)^\dagger (W_{\mu j} T_j S) + \left(S W_j^\mu T_j\right)^\dagger (S W_{\mu j} T_j)\right] \\
&= \dots \frac{g_{3L}^2}{4}(2v_1^2 + v_2^2)\left(\frac{W_8^\mu}{\sqrt{3}} W_{3\mu} + \frac{W_8^\mu}{\sqrt{3}}\frac{W_8^\mu}{\sqrt{3}} + W_3^\mu W_{3\mu} + W_3^\mu \frac{W_8^\mu}{\sqrt{3}}\right) \\
&\quad + \frac{g_{3L}^2}{2}\left[\left(v_1^2 + v_2^2\right)\left(W^{\mu+} W_\mu^- + V^{\mu+} V_\mu^-\right) + 2v_2^2 U^{\mu++} U_\mu^{--} + 2v_1 v_2 \left(W^{\mu+} V_\mu^- + V^{\mu+} W_\mu^-\right)\right] \\
&= \dots \frac{g_{3L}^2(2v_1^2 + v_2^2)}{4} \begin{pmatrix} W_3^\mu & W_8^\mu & B^\mu \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_{3\mu} \\ W_{8\mu} \\ B_\mu \end{pmatrix} \\
&\quad + \frac{g_{3L}^2}{2} \begin{pmatrix} W^{\mu+} & V^{\mu+} & U^{\mu++} \end{pmatrix} \begin{pmatrix} v_1^2 + v_2^2 & 2v_1 v_2 & 0 \\ 2v_1 v_2 & v_1^2 + v_2^2 & 0 \\ 0 & 0 & 2v_2^2 \end{pmatrix} \begin{pmatrix} W_\mu^- \\ V_\mu^- \\ U_\mu^{--} \end{pmatrix}.
\end{aligned} \tag{E.14}$$

Let us notice that in order to avoid mixing among the charged bosons, we may consider $v_1 = 0$. Moreover, this value is related with Majorana mass terms for neutrinos. In this way, the mass matrix of the charged bosons M_C, in the base $\begin{pmatrix} W_\mu & V_\mu & U_\mu \end{pmatrix}$ is

$$M_C^2 = \frac{g_{3L}^2}{2} \begin{pmatrix} v_\eta^2 + v_\rho^2 + v_2^2 & 0 & 0 \\ 0 & v_\eta^2 + v_\chi^2 + v_2^2 & 0 \\ 0 & 0 & v_\rho^2 + v_\chi^2 + 2v_2^2 \end{pmatrix}, \tag{E.15}$$

Then, masses of the charged bosons are

$$M_W^2 = \frac{g_{3L}^2}{2}\left(v_\eta^2 + v_\rho^2 + v_2^2\right), \quad M_V^2 = \frac{g_{3L}^2}{2}\left(v_\eta^2 + v_\chi^2 + v_2^2\right), \quad M_U^2 = \frac{g_{3L}^2}{2}\left(v_\rho^2 + v_\chi^2 + 2v_2^2\right). \tag{E.16}$$

The mass matrix of neutral bosons in the base $\begin{pmatrix} W_{3\mu} & W_{8\mu} & B_\mu \end{pmatrix}$ is

$$M_N^2 = \frac{g_{3L}^2}{2} \begin{pmatrix} v_\eta^2 + v_\rho^2 + v_2^2 & \frac{1}{\sqrt{3}}(v_\eta^2 - v_\rho^2 + v_2^2) & -2tv_\rho^2 \\ \frac{1}{\sqrt{3}}(v_\eta^2 - v_\rho^2 + v_2^2) & \frac{1}{3}(v_\eta^2 + v_\rho^2 + 4v_\chi^2 + v_2^2) & \frac{2t}{\sqrt{3}}(v_\rho^2 + 2v_\chi^2) \\ -2tv_\rho^2 & \frac{2t}{\sqrt{3}}(v_\rho^2 + 2v_\chi^2) & 4t^2(v_\rho^2 + v_\chi^2) \end{pmatrix}. \tag{E.17}$$

The mass eigenvalues are,

$$\begin{aligned} M_A &= 0, \\ M_Z^2 &= \frac{g_{3L}^2}{3}\left[v_\eta^2 + (1+3t^2)v_\rho^2 + (1+3t^2)v_\chi^2 + v_2^2 - \sqrt{\Delta}\right], \\ M_{Z'}^2 &= \frac{g_{3L}^2}{3}\left[v_\eta^2 + (1+3t^2)v_\rho^2 + (1+3t^2)v_\chi^2 + v_2^2 + \sqrt{\Delta}\right], \end{aligned} \tag{E.18}$$

where,

$$\Delta = v_\eta^4 + (1+3t^2)^2 v_\rho^4 + (18t^4-1)v_\rho^2 v_\chi^2 + (1+3t^2)^2 v_\chi^4 - (1+6t^2)(v_\eta^2+v_2^2)(v_\rho^2+v_\chi^2) + v_2^2(v_2^2+2v_\eta^2).$$

Now, it can be state that $v_\chi \gg v_{\eta,\rho,2}$ to define the V and U masses very large. Then using $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for $|x| \ll 1$,

$$\begin{aligned} \sqrt{\Delta} &\approx v_\chi^2(1+3t^2)\sqrt{1 + \left[\frac{18t^4-1}{(1+3t^2)^2}\right]\frac{v_\rho^2}{v_\chi^2} - \left[\frac{1+6t^2}{(1+3t^2)^2}\right]\left(\frac{v_\eta^2+v_2^2}{v_\chi^2}\right)} \\ &\approx (1+3t^2)v_\chi^2 + \left(\frac{18t^4-1}{1+3t^2}\right)\frac{v_\rho^2}{2} - \left(\frac{1+6t^2}{1+3t^2}\right)\frac{v_\eta^2+v_2^2}{2}. \end{aligned} \tag{E.19}$$

Then, neutral bosons have the following masses

$$M_A = 0,\ M_Z^2 \approx \frac{g_{3L}^2}{2}\left(\frac{1+4t^2}{1+3t^2}\right)(v_\eta^2+v_\rho^2+v_2^2),\ M_{Z'}^2 \approx \frac{2g_{3L}^2}{3}\left(1+3t^2\right)v_\chi^2. \tag{E.20}$$

The W/Z mass ratio compared with (1.35) allows us to obtain the $t = \frac{g_N}{g_{3L}}$ value.

$$\frac{M_W^2}{M_Z^2} = \frac{1+3t^2}{1+4t^2} = \cos^2\theta_W, \tag{E.21}$$

then the value of t is known,

$$t^2 = \frac{\sin^2\theta_W}{1-4\sin^2\theta_W}. \tag{E.22}$$

Let us go back to the matrix analysis in (E.17) with the same assumptions $v_\chi \gg v_{\eta,\rho,2}$. The eigenvectors are

$$
\begin{aligned}
V_A &= \begin{pmatrix} t \\ -\sqrt{3}t \\ 1 \end{pmatrix}, \\
V_Z &= \begin{pmatrix} \frac{3t^2(v_\eta^2 v_\rho^2 + v_\eta^2 v_\chi^2 + v_\rho^2 v_\chi^2 + v_2^2 v_\rho^2 - v_\rho^4 + v_2^2 v_\chi^2) + \sqrt{\Delta}(v_\eta^2 + v_\rho^2 + v_2^2) + v_\eta^2 v_\chi^2 - 2v_2^2 v_\eta^2 - v_\eta^4 + v_\rho^2 v_\chi^2 - v_\rho^4 + v_2^2 v_\chi^2 - v_2^4}{2t\left(3t^2 v_\rho^2 v_\chi^2 + 3t^2 v_\rho^4 - \sqrt{\Delta} v_\rho^2 - v_\eta^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2\right)} \\ \frac{3t^2(v_\eta^2 v_\rho^2 + v_\eta^2 v_\chi^2 + 3v_\rho^2 v_\chi^2 + v_2^2 v_\rho^2 + v_\rho^4 + v_2^2 v_\chi^2) + \sqrt{\Delta}(v_\eta^2 - v_\rho^2 + v_2^2) - v_\eta^2 v_\chi^2 - 2v_2^2 v_\eta^2 - v_\eta^4 + v_\rho^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2 - v_2^4}{2\sqrt{3}t\left(3t^2 v_\rho^2 v_\chi^2 + 3t^2 v_\rho^4 - \sqrt{\Delta} v_\rho^2 - v_\eta^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2\right)} \\ 1 \end{pmatrix}, \\
V_{Z'} &= \begin{pmatrix} -\frac{-3t^2(v_\eta^2 v_\rho^2 + v_\eta^2 v_\chi^2 + v_\rho^2 v_\chi^2 + v_2^2 v_\rho^2 - v_\rho^4 + v_2^2 v_\chi^2) + \sqrt{\Delta}(v_\eta^2 + v_\rho^2 + v_2^2) - v_\eta^2 v_\chi^2 + 2v_2^2 v_\eta^2 + v_\eta^4 - v_\rho^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2 + v_2^4}{2t\left(3t^2 v_\rho^2 v_\chi^2 + 3t^2 v_\rho^4 + \sqrt{\Delta} v_\rho^2 - v_\eta^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2\right)} \\ -\frac{-3t^2(v_\eta^2 v_\rho^2 + v_\eta^2 v_\chi^2 + 3v_\rho^2 v_\chi^2 + v_2^2 v_\rho^2 + v_\rho^4 + v_2^2 v_\chi^2) + \sqrt{\Delta}(v_\eta^2 - v_\rho^2 + v_2^2) + v_\eta^2 v_\chi^2 + 2v_2^2 v_\eta^2 + v_\eta^4 - v_\rho^2 v_\chi^2 - v_\rho^4 + v_2^2 v_\chi^2 + v_2^4}{2\sqrt{3}t\left(3t^2 v_\rho^2 v_\chi^2 + 3t^2 v_\rho^4 + \sqrt{\Delta} v_\rho^2 - v_\eta^2 v_\chi^2 + v_\rho^4 - v_2^2 v_\chi^2\right)} \\ 1 \end{pmatrix}.
\end{aligned}
$$

Now, the approximation for $\sqrt{\Delta}$ computed in (E.19) can be used as well as a simple normalization,

$$
V_A = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} t \\ -\sqrt{3}t \\ 1 \end{pmatrix}, \quad V_Z \approx \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} \sqrt{1+3t^2} \\ \frac{\sqrt{3}t^2}{\sqrt{1+3t^2}} \\ -\frac{t}{\sqrt{1+3t^2}} \end{pmatrix}, \quad V_{Z'} \approx \frac{1}{\sqrt{1+3t^2}} \begin{pmatrix} 0 \\ 1 \\ \sqrt{3}t \end{pmatrix}. \tag{E.23}
$$

So, the mass eigenstates for the neutral vector bosons are

$$
\begin{aligned}
A^\mu &= \frac{1}{\sqrt{1+4t^2}} \left[\left(W_3^\mu - \sqrt{3}\, W_8^\mu \right) t + B^\mu \right] \\
Z^\mu &\approx \frac{1}{\sqrt{1+4t^2}} \left(\sqrt{1+3t^2}\, W_3^\mu + \frac{\sqrt{3}t^2}{\sqrt{1+3t^2}}\, W_8^\mu - \frac{t}{\sqrt{1+3t^2}}\, B^\mu \right) \\
Z^{\mu'} &\approx \frac{1}{\sqrt{1+3t^2}} \left(W_8^\mu + \sqrt{3}t\, B^\mu \right)
\end{aligned} \tag{E.24}
$$

Solving for the gauge bosons:

$$\begin{aligned}
W_{3\mu} &\approx \frac{1}{\sqrt{1+4t^2}}\left(A_\mu t + Z_\mu\sqrt{1+3t^2}\right)\\
W_{8\mu} &\approx \frac{1}{\sqrt{1+3t^2}}\left(-A_\mu t\sqrt{\frac{3+9t^2}{1+4t^2}} + Z_\mu t^2\sqrt{\frac{3}{1+4t^2}} + Z'_\mu\right)\\
B_\mu &\approx \frac{1}{\sqrt{1+4t^2}}\left(A_\mu - Z_\mu t\sqrt{\frac{1}{1+3t^2}} + Z'_\mu t\sqrt{\frac{3+12t^2}{1+3t^2}}\right)
\end{aligned} \tag{E.25}$$

As in the SM case, we want to get the electromagnetic current from the breakdown of the leptonic sector,

$$\begin{aligned}
i\left[\bar{L}_l \not{D} L_l\right] &= i\left[\bar{L}\gamma^\mu\left(\partial_\mu + ig_{3L}W_{\mu j}T_j\right)L\right]\\
&= \ldots - \frac{g_{3L}}{2}\bar{e}_L\gamma^\mu\left(-W_{3\mu} + \frac{W_{8\mu}}{\sqrt{3}}\right)e_L + \ldots\\
&= \ldots - \frac{g_{3L}}{2}\bar{e}_L\gamma^\mu\left(-\frac{A_\mu t}{\sqrt{1+4t^2}} - \frac{A_\mu t}{\sqrt{1+4t^2}} + \ldots\right)e_L + \ldots\\
&= \ldots + g_{3L}\bar{e}_L\gamma^\mu\left(\frac{t}{\sqrt{1+4t^2}}\right)A_\mu e_L + \ldots.
\end{aligned} \tag{E.26}$$

If we define $t = \tan\theta$, then the relation between electric charge and couplings are

$$e = g_{3L}\left(\frac{t}{\sqrt{1+4t^2}}\right) = \frac{g_{3L}\sin\theta}{\sqrt{1+3\sin^2\theta}} = \frac{g_N\cos\theta}{\sqrt{1+3\sin^2\theta}} \tag{E.27}$$

From the last equation,

$$e = \frac{g_N g_{3L}}{\sqrt{g_{3L}^2 + 4g_N^2}} \rightarrow \boxed{\frac{1}{e^2} = \frac{4}{g_{3L}^2} + \frac{1}{g_N^2}} \tag{E.28}$$

E.3 Extension: 221-LR symmetry

Using the same logic as the previous cases, in this symmetry the covariant derivative for scalar triplets are

$$D_\mu\Delta_{L,R} = \partial_\mu\Delta_{L,R} + ig\left[\widetilde{W}_{\mu\,L,R}, \Delta_{L,R}\right] + ig_{BL}\Delta_{L,R}, \tag{E.29}$$

where it has been used $g_L = g_R = g$ and the hypercharge $B-L = +2$ which was defined in the Lagrangian due to the lepton couplings.

The eigenstates for the charged (complex) vector bosons are $W^\pm_{\mu\,L,R} = \left[\frac{1}{\sqrt{2}}\left(W_{1\mu} \mp iW_{2\mu}\right)\right]_{L,R}$

and the vector boson matrix is

$$\widetilde{W}_{\mu\,L,R} = \begin{pmatrix} \dfrac{W_{3\mu}}{2} & \dfrac{W_\mu^+}{\sqrt{2}} \\ \dfrac{W_\mu^-}{\sqrt{2}} & -\dfrac{W_{3\mu}}{2} \end{pmatrix}_{L,R} .$$

The vector boson mass matrices are built from

$$\mathrm{Tr}\left[(D_\mu\Delta_L)^\dagger\,(D^\mu\Delta_L)\right] + \mathrm{Tr}\left[(D_\mu\Delta_R)^\dagger\,(D^\mu\Delta_R)\right],$$

with the vev's defined as [43]

$$\langle\Delta_{L,R}\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ v_{L,R} & 0 \end{pmatrix}, \quad \text{and} \quad \langle\Phi\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}. \tag{E.30}$$

For $\Delta_a,\ a = L, R$ the commutator $\left[\widetilde{W}_{\mu\,a}, \Delta_a\right] = \frac{v_a}{2}\begin{pmatrix} W_\mu^+ & 0 \\ -\frac{2}{\sqrt{2}}W_{3\mu} & -W_\mu^+ \end{pmatrix}$, then

$$\begin{aligned}
\mathrm{Tr}\left[(D_\mu\Delta_a)^\dagger\,(D^\mu\Delta_a)\right] &= \ldots\,\mathrm{Tr}\left\{\left(\partial_\mu\Delta_a + ig\left[\widetilde{W}_\mu, \Delta_a\right] + igB_\mu\Delta_a\right)^\dagger\left(\partial^\mu\Delta_a + ig\left[\widetilde{W}^\mu, \Delta_a\right] + igB^\mu\Delta_a\right)\right\} \\
&= \ldots\,\frac{g^2v_a^2}{2}\left(W_{a\mu}^- W_a^{\mu+} + W_{3a\mu}W_{3a}^\mu\right) - \frac{g\,g_{BL}v_a^2}{2}\left(W_{3a\mu}B^\mu + B_\mu W_{3a}^\mu\right) + \frac{g_{BL}^2 v_a^2}{2}B_\mu B^\mu \\
&= \ldots\begin{pmatrix} W_{3\mu L} & W_{3\mu R} & B_\mu \end{pmatrix}\begin{pmatrix} \frac{1}{2}g^2v_L^2 & 0 & -\frac{1}{2}g\,g_{BL}v_L^2 \\ 0 & \frac{1}{2}g^2v_R^2 & -\frac{1}{2}g\,g_{BL}v_R^2 \\ -\frac{1}{2}g\,g_{BL}v_L^2 & -\frac{1}{2}g\,g_{BL}v_R^2 & \frac{1}{2}g_{BL}^2(v_L^2+v_R^2) \end{pmatrix}\begin{pmatrix} W_{3L}^\mu \\ W_{3R}^\mu \\ B^\mu \end{pmatrix} \\
&\quad + \begin{pmatrix} W_{\mu L}^- & W_{\mu R}^- \end{pmatrix}\begin{pmatrix} \frac{1}{2}g^2v_L^2 & 0 \\ 0 & \frac{1}{2}g^2v_R^2 \end{pmatrix}\begin{pmatrix} W_{\mu L}^+ \\ W_{\mu R}^+ \end{pmatrix}.
\end{aligned} \tag{E.31}$$

The covariant derivative for the bidoublet is

$$D_\mu\Phi = \partial_\Phi + ig\left(\widetilde{W}_{\mu L}\Phi - \Phi\widetilde{W}_{\mu R}\right), \tag{E.32}$$

then

$$\begin{aligned}
&\mathrm{Tr}\left[(D_\mu\Phi)^\dagger\,(D^\mu\Phi)\right] = (\partial_\mu\Phi)^\dagger\,(\partial^\mu\Phi) + \ldots \\
&g^2\,\mathrm{Tr}\left\{\begin{pmatrix} \frac{k}{2\sqrt{2}}(W_{3\mu L}-W_{3\mu R}) & \frac{k}{2}W^+_{\mu L} - \frac{k}{2}'W^+_{\mu R} \\ \frac{k}{2}'W^-_{\mu L} - \frac{k}{2}W^-_{\mu R} & -\frac{k'}{2\sqrt{2}}(W_{3\mu L}-W_{3\mu R}) \end{pmatrix}\begin{pmatrix} \frac{k}{2\sqrt{2}}(W^\mu_{3L}-W^\mu_{3R}) & \frac{k}{2}'W^{\mu+}_{L} - \frac{k}{2}W^{\mu+}_{R} \\ \frac{k}{2}W^{\mu-}_{L} - \frac{k'}{2}W^{\mu-}_{R} & -\frac{k'}{2\sqrt{2}}(W^\mu_{3L}-W^\mu_{3R}) \end{pmatrix}\right\} \\
&= \ldots g^2\left[\frac{k^2}{8}\left(W_{3\mu L}W^\mu_{3L} - W_{3\mu L}W^\mu_{3R} - W_{3\mu R}W^\mu_{3L} + W_{3\mu R}W^\mu_{3R}\right) + \frac{k^2}{4}W^+_{\mu L}W^{\mu-}_{R} - \frac{kk'}{4}W^+_{\mu L}W^{\mu-}_{R}\right. \\
&\quad -\frac{kk'}{4}W^+_{\mu R}W^{\mu-}_{L} + \frac{k'^2}{4}W^-_{\mu L}W^{\mu+}_{L} - \frac{kk'}{4}W^-_{\mu L}W^{\mu+}_{R} - \frac{kk'}{4}W^-_{\mu R}W^{\mu+}_{L} + \frac{k^2}{4}W^-_{\mu R}W^{\mu+}_{R} \\
&\quad \left. + \frac{k'^2}{8}\left(W_{3\mu L}W^\mu_{3L} - W_{3\mu L}W^\mu_{3R} - W_{3\mu R}W^\mu_{3L} + W_{3\mu R}W^\mu_{3R}\right)\right] \\
&= \ldots \begin{pmatrix} W_{3\mu L} & W_{3\mu R} & B_\mu \end{pmatrix}\begin{pmatrix} \frac{1}{8}g^2(k^2+k'^2) & -\frac{1}{8}g^2(k^2+k'^2) & 0 \\ -\frac{1}{8}g^2(k^2+k'^2) & \frac{1}{8}g^2(k^2+k'^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} W^\mu_{3L} \\ W^\mu_{3R} \\ B^\mu \end{pmatrix} \\
&\quad + \begin{pmatrix} W^-_{\mu L} & W^-_{\mu R} \end{pmatrix}\begin{pmatrix} \frac{g^2}{4}(k^2+k'^2) & -\frac{g^2}{2}kk' \\ -\frac{g^2}{2}kk' & \frac{g^2}{4}(k^2+k'^2) \end{pmatrix}\begin{pmatrix} W^+_{\mu L} \\ W^+_{\mu R} \end{pmatrix}.
\end{aligned} \tag{E.33}$$

The mass matrix of charged vector bosons in the base $\begin{pmatrix} W^-_{\mu L} & W^-_{\mu R} \end{pmatrix}$ is

$$M^2_C = \frac{g^2}{4}\begin{pmatrix} 2v_L^2 + k^2 + k'^2 & -2kk' \\ -2kk' & 2v_R^2 + k^2 + k'^2 \end{pmatrix}, \tag{E.34}$$

and the eigenvalues are

$$M^2_{1,2} = \frac{g^2}{4}\left(k^2 + k'^2 + v_L^2 + v_R^2 \mp \sqrt{(v_R^2 - v_L^2)^2 + 4k^2k'^2}\right). \tag{E.35}$$

These eigenvalues are related with the mass eigenstates. Assuming $v_R \gg k, k' \gg v_L$, and $k_+^2 \equiv k^2 + k'^2$

$$\begin{aligned}
M^2_{W_1} &= \frac{g^2}{4}\left[k_+^2 + v_R^2\left(1 + \frac{v_L^2}{v_R^2}\right) - v_R^2\sqrt{1 - 2\frac{v_L^2}{v_R^2} + \frac{v_L^4}{v_R^4} + 4\frac{k^2k'^2}{v_R^4}}\right] \\
&\approx \frac{g^2}{4}\left[k_+^2 + v_R^2 - v_R^2\sqrt{1 + 4\frac{k^2k'^2}{v_R^4}}\right] \\
&\approx \frac{g^2}{4}\left[k_+^2 + v_R^2 - v_R^2\left(1 + 2\frac{k^2k'^2}{v_R^4}\right)\right] \\
&\approx \boxed{\frac{g^2}{4}k_+^2\left(1 - 2\frac{k^2k'^2}{v_R^2k_+^2}\right)}
\end{aligned} \tag{E.36}$$

$$\begin{aligned} M_{W_2}^2 &= \frac{g^2}{4}\left[k_+^2 + v_R^2\left(1+\frac{v_L^2}{v_R^2}\right) + v_R^2\sqrt{1-2\frac{v_L^2}{v_R^2}+\frac{v_L^4}{v_R^4}+4\frac{k^2k'^2}{v_R^4}}\right] \\ &\approx \frac{g^2}{4}\left[k_+^2 + v_R^2 + v_R^2\sqrt{1+4\frac{k^2k'^2}{v_R^4}}\right] \\ &\approx \frac{g^2}{4}v_R^2\left[2+\frac{k_+^2}{v_R^2}+2\frac{k^2}{v_R^2}\frac{k'^2}{v_R^2}\right] \\ &\approx \boxed{\frac{g^2}{2}v_R^2\left(1+\frac{k_+^2}{2v_R^2}\right)} \end{aligned} \tag{E.37}$$

The mass matrix for neutral bosons in the base $\begin{pmatrix} W_{3\mu L} & W_{3\mu R} & B_\mu \end{pmatrix}$ is

$$M_N^2 = \begin{pmatrix} \frac{1}{4}g^2(4v_L^2+k_+^2) & -\frac{1}{4}g^2k_+^2 & -gg_{BL}v_L^2 \\ -\frac{1}{4}g^2k_+^2 & \frac{1}{4}g^2(v_R^+k_+^2) & -gg_{BL}v_R^2 \\ -gg_{BL}v_L^2 & -gg_{BL}v_R^2 & g_{BL}^2(v_L^2+v_R^2), \end{pmatrix} \tag{E.38}$$

where $k_+^2 = k^2 + k'^2$.

The mass eigenvalues in the $v_L \to 0$ limit for Z_1 and Z_2 (for A is exact) is

$$\boxed{M_A = 0} \tag{E.39}$$

$$\begin{aligned} M_{Z_1}^2 &= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) - \sqrt{g^4k_+^4 + 4v_R^4(g^2+g_{BL}^2)^2 - 4k_+^2v_R^2g^2g_{BL}^2}\right] \\ &= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) - 2v_R^2(g^2+g_{BL}^2)\sqrt{1+\frac{g^4k_+^4}{4v_R^4(g^2+g_{BL}^2)^2} - \frac{g^2g_{BL}^2k_+^2}{v_R^2(k_+^2v_R^2)^2}}\right] \\ &\approx \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) - 2v_R^2(g^2+g_{BL}^2)\left(1-\frac{g^2g_{BL}^2k_+^2}{2v_R^2(g^2+g_{BL}^2)^2}+\frac{g^6k_+^4(g^2+2g_{BL}^2)}{8v_R^4(g^2+g_{BL}^2)^4}\right)\right] \\ &= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2)\left(\frac{g^2g_{BL}^2k_+^2}{2v_R^2(g^2+g_{BL}^2)^2}-\frac{g^6k_+^4(g^2+2g_{BL}^2)}{8v_R^4(g^2+g_{BL}^2)^4}\right)\right] \\ &= \frac{1}{4}\left[g^2k_+^2 + k_+^2g^2\left(\frac{g_{BL}^2}{g^2+g_{BL}^2}-\frac{g^4k_+^2(g^2+2g_{BL}^2)}{4v_R^4(g^2+g_{BL}^2)^3}\right)\right] \\ &= \frac{g^2k_+^2}{4}\left[\frac{g^2+2g_{BL}^2}{g^2+g_{BL}^2}-\frac{g^4k_+^2}{4v_R^2}\frac{g^2+2g_{BL}^2}{(g^2+g_{BL}^2)^3}\right] \\ &= \frac{g^2k_+^2}{4}\left(\frac{g^2+2g_{BL}^2}{g^2+g_{BL}^2}\right)\left[1-\frac{k_+^2}{4v_R^2}\left(\frac{g^4}{(g^2+g_{BL}^2)^2}\right)\right] \\ &= \boxed{\frac{g^2k_+^2}{4\cos^2\theta}\left(1-\frac{k_+^2}{4v_R^2\cos^4\theta_{BL}}\right)}, \end{aligned} \tag{E.40}$$

where the phases θ and θ_{BL} are defined as

$$\sin\theta = \frac{g_{BL}}{\sqrt{g^2+2g_{BL}^2}}, \quad \text{and} \quad \sin\theta_{BL} = \frac{g_{BL}}{\sqrt{g^2+g_{BL}^2}} \tag{E.41}$$

The third mass eigenvalue is:

$$\begin{aligned}
M_{Z_2}^2 &= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) + \sqrt{g^4k_+^4 + 4v_R^4(g^2+g_{BL}^2)^2 - 4k_+^2v_R^2g^2g_{BL}^2}\right] \\
&= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) + 2v_R^2(g^2+g_{BL}^2)\sqrt{1 + \frac{g^4k_+^4}{4v_R^4(g^2+g_{BL}^2)^2} - \frac{g^2g_{BL}^2k_+^2}{v_R^2(k_+^2v_R^2)^2}}\right] \\
&\approx \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2) + 2v_R^2(g^2+g_{BL}^2)\left(1 - \frac{g^2g_{BL}^2k_+^2}{2v_R^2(g^2+g_{BL}^2)^2} + \frac{g^6k_+^4(g^2+2g_{BL}^2)}{8v_R^4(g^2+g_{BL}^2)^4}\right)\right] \\
&= \frac{1}{4}\left[g^2k_+^2 + 2v_R^2(g^2+g_{BL}^2)\left(2 - \frac{g^2g_{BL}^2k_+^2}{2v_R^2(g^2+g_{BL}^2)^2} + \frac{g^6k_+^4(g^2+2g_{BL}^2)}{8v_R^4(g^2+g_{BL}^2)^4}\right)\right] \\
&= \frac{1}{4}\left[4v_R^2(g^2+g_{BL}^2) + g^2k_+^2\left(1 - \frac{g_{BL}^2}{g^2+g_{BL}^2}\right)\right] \\
&= \boxed{v_R^2(g^2+g_{BL}^2)\left[1 + \frac{k_+^2\cos^4\theta_{BL}}{4v_R^2}\right]}
\end{aligned} \tag{E.42}$$

The mass eigenstates must include those of the SM, en this sense we may relate $W_1 \to W$ and $Z_1 \to Z$. Also, we obtained new vector bosons that may be associated to very large masses in comparison to SM ones, $W_2 \to W_R$ and $Z_2 \to Z'$.

Then the mass ratio for the known SM vector bosons, equating (E.5) with (E.36) and (E.40),

$$\frac{M_W^2}{M_Z^2} \overset{\text{SM}}{=} \frac{g^2}{g^2+g_Y^2} \overset{\text{LR}}{\approx} \frac{\frac{g^2k_+^2}{4}}{\frac{g^2k_+^2}{4\cos^2\theta}} = \cos^2\theta. \tag{E.43}$$

Then $\cos^2\theta_W = \cos^2\theta$, that is

$$\frac{g^2}{g^2+g_Y^2} = \frac{g^2+g_{BL}^2}{g^2+2g_{BL}^2} \Rightarrow \frac{1}{g_Y^2} = \frac{1}{g^2} + \frac{1}{g_{BL}^2}. \tag{E.44}$$

Finally, from (E.8),

$$\boxed{\frac{1}{e^2} \approx \frac{2}{g^2} + \frac{1}{g_{BL}^2}} \tag{E.45}$$

Appendix F

Covariant derivative in $\mathrm{SU(N)}$

All things we are going to show are based on the requirement that the covariant derivative must transform like the field it acts to.

$$\Phi \to \Phi' = U\Phi \Rightarrow D_\mu\Phi \to (D_\mu\Phi)' = UD_\mu\Phi, \tag{F.1}$$

where U is the matrix transformation (or scalar in the Abelian case) for the corresponding Lie group.

F.1 Abelian case

$$\begin{aligned} \phi' &= e^{ig\theta(x)}\phi \\ (D_\mu\phi)' &= e^{ig\theta(x)}D_\mu\phi, \end{aligned} \tag{F.2}$$

where $\theta(x)$ is a real function. It is easy to show that the requirement is achieved with:

$$D_\mu\phi = \partial_\mu\phi + igA_\mu\phi, \text{ and } A'_\mu = A_\mu - \partial_\mu\theta(x), \tag{F.3}$$

being A_μ its corresponding gauge field.

In the infinitesimal version, for small $\theta \to \delta\theta$ then $\phi \to (1 + ig\delta\theta)\phi$,

$$\delta\phi = ig\phi\delta\theta, \ \delta D_\mu\phi = igD_\mu\phi\delta\theta, \ \delta A_\mu = -\partial_\mu(\delta\theta). \tag{F.4}$$

On the other hand, the antisymmetric electromagnetic field tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant,

$$\delta F_{\mu\nu} = \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu) = -\partial_\mu\partial_\nu(\delta\theta) + \partial_\nu\partial_\mu(\delta\theta) = 0.$$

The latter means that $F_{\mu\nu}\psi \to (F_{\mu\nu}\psi)' = U(F_{\mu\nu}\psi)$ and the Kinetic term of the Lagrangian for gauge bosons, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, that contains a quadratic term in $\partial_\mu A_\nu$, is gauge invariant.

Besides, $[D_\mu, D_\nu]\psi = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]\psi = ig\left([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]\right)\psi = igF_{\mu\nu}\psi$. then,

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu], \tag{F.5}$$

F.2 Non-Abelian case

F.2.1 Gauge transformation of fields in matrix representation

In the non-Abelian case, the transformations depend upon the representations; if the field has a column representation: $\Phi \to \Phi' = U\Phi$, or if it is row: $\Phi^\dagger \to \Phi'^\dagger = \Phi^\dagger U^\dagger$. Besides, there some gauge invariants such as $g_{\mu\nu} \to g_{\mu\nu}$ and the scalar

$$\left(\Phi^\dagger\Phi\right) \to \left(\Phi^\dagger\Phi\right)' = \Phi^\dagger\Phi. \tag{F.6}$$

For square matrices $M_{1,2}$ that are coupled with fields in different ways,

$$\begin{aligned} \left(\Phi^\dagger M_1\Phi\right)' = &\quad \Phi^\dagger U^\dagger M_1' U\Phi \quad \Rightarrow \quad M_1' = UM_1U^\dagger \text{ and } M_1'^* = U^*M_1^*U^T, \\ \left(\Phi^T M_2\Phi\right)' = &\quad \Phi^T U^T M_2' U\Phi \quad \Rightarrow \quad M_2' = U^*M_2U^\dagger \text{ and } M_2'^* = UM_2^*U^T. \end{aligned} \tag{F.7}$$

F.2.2 Properties of the gauge covariant derivative

From the requirement raised at the beginning $(D_\mu\Phi)' = D_\mu'\Phi' = D_\mu'(U\Phi) \to UD_\mu\Phi$. Then

$$\boxed{D_\mu' = UD_\mu U^\dagger}. \tag{F.8}$$

In general, gauge covariant derivatives are defined to satisfy the Leibniz rule (the product rule) and to reduce to the ordinary partial derivative when applied to an invariant scalar object.

$$D_\mu\left(\Psi^\dagger\Phi\right) = \left(D_\mu\Psi^\dagger\right)\Phi + \Psi^\dagger D_\mu\Phi = \partial_\mu\left(\Psi^\dagger\Phi\right)$$
$$D_\mu\left(\Phi^\dagger\Psi\right) = \left(D_\mu\Phi^\dagger\right)\Psi + \Phi^\dagger D_\mu\Psi = \partial_\mu\left(\Phi^\dagger\Psi\right),$$

then $\Psi^\dagger\left(D_\mu\Phi^\dagger\right)^\dagger + \left(D_\mu\Psi\right)^\dagger\Phi = \Psi^\dagger\left(D_\mu\Phi\right) + (D_\mu\Psi^\dagger)\Phi$. In other words,

$$\left(D_\mu\Phi^\dagger\right)^\dagger = D_\mu\Phi \text{ or } \left(D_\mu\Phi\right)^\dagger = D_\mu\Phi^\dagger, \tag{F.9}$$

and the same relation with Ψ.

The conjugate transpose commutes with the gauge covariant derivative. If $F(\Psi) = \Psi^\dagger \Rightarrow [F, D_\mu] = 0$.

F.2.3 Construction of the gauge covariant derivative

Ansatz: $D_\mu = \mathbf{1}\partial_\mu + igA_\mu$; being $A_\mu \rightarrow$ square matrix of the same dimension as the transformation matrix U. Since the transformation is $D'_\mu = \mathbf{1}\partial_\mu + igA'_\mu = U\left(\mathbf{1}\partial_\mu + igA_\mu\right)U^\dagger$, then

$$\boxed{A'_\mu = \frac{i}{g}(\partial_\mu U)U^\dagger + UA_\mu U^\dagger} \tag{F.10}$$

Property: If there exists a field B_μ that commutes with U, then $(A_\mu + B_\mu)' = i(\partial_\mu U)U^\dagger + U(A_\mu + B_\mu)U^\dagger = A'_\mu$ which means that B_μ is a gauge invariant, $B_\mu \rightarrow B'_\mu = B_\mu$.

Also, we can clear the form of the covariant derivative for transpose conjugates $D_\mu\Psi^\dagger$ from the requirement

$$\begin{aligned}\partial_\mu(\Psi^\dagger\Phi) = D_\mu(\Psi^\dagger\Phi) \Rightarrow (\partial_\mu\Psi^\dagger)\Phi + \Psi^\dagger(\partial_\mu\Phi) &= (D_\mu\Psi^\dagger)\Phi + \Psi^\dagger(D_\mu\Phi)\\ &= (D_\mu\Psi^\dagger)\Phi + \Psi^\dagger(\partial_\mu\Phi + igA_\mu\Phi)\\ \Rightarrow (D_\mu\Psi^\dagger)\Phi &= (\partial_\mu\Psi^\dagger)\Phi - ig\Psi^\dagger A_\mu\Phi,\end{aligned}$$

with which we obtain the covariant derivative for the fields $\Psi^\dagger$:

$$\boxed{D_\mu\Psi^\dagger = \partial_\mu\Psi^\dagger - ig\Psi^\dagger A_\mu \quad \text{or} \quad D_\mu\Psi^T = \partial_\mu\Psi^T + ig\Psi^T A^T_\mu}. \tag{F.11}$$

In order to be in accordance with (F.9),

$$
\begin{aligned}
(D_\mu \Psi)^\dagger = D_\mu \psi^\dagger \Rightarrow (\partial_\mu \Psi + igA_\mu \Psi)^\dagger &= \partial_\mu \Psi^\dagger - ig\Psi^\dagger A_\mu \\
\Rightarrow \boxed{A_\mu^\dagger = A_\mu}.
\end{aligned}
\tag{F.12}
$$

Now, let be $M_1 = \Phi\Psi^\dagger$ and $M_2 = \Phi\Psi^T$ square matrices (outer product), with $\Phi' = U\Phi$ and $\Psi' = V\Psi$, the transformations may be written down as $M_1' \to UM_1'V^\dagger$ and $M_2' \to UM_2'V^T$. Then,

$$
D_\mu(\Phi\Psi^\dagger) = \left(\partial_\mu \Phi + igA_\mu^U \Phi\right)\Psi^\dagger + \Phi\left(\partial_\mu \Psi^\dagger - ig\Psi^\dagger A_\mu^V\right) = \partial_\mu(\Phi\Psi^\dagger) + ig\left(A_\mu^U \Phi\Psi^\dagger - \Phi\Psi^\dagger A_\mu^V\right),
$$

$$
D_\mu(\Phi\Psi^T) = \left(\partial_\mu \Phi + igA_\mu^U \Phi\right)\Psi^T + \Phi\left(\partial_\mu \Psi^T + ig\Psi^T A_\mu^{*\,U}\right) = \partial_\mu(\Phi\Psi^T) + i\{A_\mu^U \Phi\Psi^T + \Phi\Psi^T A_\mu^{*\,V}\},
$$

then,

$$
\boxed{D_\mu M_1 = \partial_\mu M_1 + ig\left(A_\mu^U M_1 - M_1 A_\mu^V\right)} \quad \text{and} \quad \boxed{D_\mu M_2 = \partial_\mu M_2 + ig\{A_\mu^U M_2 + M_2 A_\mu^{*\,V}\}}. \tag{F.13}
$$

Furthermore, if $M_1 = \Phi\Phi^\dagger$ then it is a Hermitian matrix that transforms as $UM_1U^\dagger$. In the same way, if $M_2 = \Psi\Psi^T$ then it is a symmetric matrix and transformas as UM_2U^T. Both multiplets have their covariant derivatives as

$$
D_\mu M_1 = \partial_\mu M_1 + ig\left[A_\mu^U, M_1\right] \quad \text{and} \quad D_\mu M_2 = \partial_\mu M_2 + ig\{A_\mu M_2 + M_2 A_\mu^*\}. \tag{F.14}
$$

Bibliography

[1] P. D. Group, P. A. Zyla, , *et al.*, "Review of particle physics," *Progress of Theoretical and Experimental Physics*, vol. 2020, Aug 2020. 083C01.

[2] H. Politzer, "Reliable Perturbative Results for Strong Interactions?," *Phys. Rev. Lett.*, vol. 30, pp. 1346–1349, 1973.

[3] D. J. Gross and F. Wilczek, "Ultraviolet Behavior of Nonabelian Gauge Theories," *Phys. Rev. Lett.*, vol. 30, pp. 1343–1346, 1973.

[4] S. L. Glashow, "Partial Symmetries of Weak Interactions," *Nucl. Phys.*, vol. 22, pp. 579–588, 1961.

[5] S. Weinberg, "A model of leptons," *Phys. Rev. Lett.*, vol. 19, pp. 1264–1266, Nov 1967.

[6] A. Salam, "Weak and Electromagnetic Interactions," *Conf. Proc. C*, vol. 680519, pp. 367–377, 1968.

[7] P. W. Higgs, "Broken symmetries and the masses of gauge bosons," *Phys. Rev. Lett.*, vol. 13, pp. 508–509, Oct 1964.

[8] J. Goldstone, A. Salam, and S. Weinberg, "Broken symmetries," *Phys. Rev.*, vol. 127, pp. 965–970, Aug 1962.

[9] M. Reig, J. W. F. Valle, and C. A. Vaquera-Araujo, "Unifying left–right symmetry and 331 electroweak theories," *Phys. Lett.*, vol. B766, pp. 35–40, 2017.

[10] E. T. Franco and V. Pleitez, "Left-right symmetric extensions of 3-3-1 models," 2017.

[11] D. Borah and S. Patra, "Universal seesaw and $0\nu\beta\beta$ in new 3331 left-right symmetric model," *Phys. Lett.*, vol. B771, pp. 318–326, 2017.

[12] A. G. Dias, C. A. de S. Pires, and P. S. Rodrigues da Silva, "The Left-Right $SU(3)_L \otimes SU(3)_R \otimes U(1)_X$ Model with Light, keV and Heavy Neutrinos," *Phys. Rev.*, vol. D82, p. 035013, 2010.

[13] M. Gell-Mann, "The interpretation of the new particles as displaced charge multiplets," *Il Nuovo Cimento (1955-1965)*, vol. 4, no. 2, pp. 848–866, 1956.

[14] R. J. Finkelstein, "On the quantization of a unitary field theory," *Phys. Rev.*, vol. 75, pp. 1079–1087, Apr 1949.

[15] T. Rashid, *Development of pixel detector for ATLAS Inner Tracker(ITK) upgrade at HL-LHC and Searching for the Standard Model Higgs boson decay into b-quark pair with ATLAS experiment.* PhD thesis, Université Paris Saclay (COmUE), 01 2019.

[16] M. Ibe, A. Kusenko, and T. T. Yanagida, "Why three generations?," *Physics Letters B*, vol. 758, no. Supplement C, pp. 365 – 369, 2016.

[17] H. Georgi and S. L. Glashow, "Gauge theories without anomalies," *Phys. Rev. D*, vol. 6, pp. 429–431, Jul 1972.

[18] J. L. Rosner, "Resource letter: Sm-1: The standard model and beyond," *American Journal of Physics*, vol. 71, no. 4, pp. 302–318, 2003.

[19] T. D. Lee, "A theory of spontaneous t violation," *Phys. Rev. D*, vol. 8, pp. 1226–1239, Aug 1973.

[20] H. E. Haber and G. L. Kane, "The Search for Supersymmetry: Probing Physics Beyond the Standard Model," *Phys. Rept.*, vol. 117, pp. 75–263, 1985.

[21] J. E. Kim, "Light Pseudoscalars, Particle Physics and Cosmology," *Phys. Rept.*, vol. 150, pp. 1–177, 1987.

[22] D. A. Camargo, M. D. Campos, T. B. de Melo, and F. S. Queiroz, "A Two Higgs Doublet Model for Dark Matter and Neutrino Masses," *Phys. Lett. B*, vol. 795, pp. 319–326, 2019.

[23] S. Bahrami and M. Frank, "Dark Matter in the Higgs Triplet Model," *Phys. Rev. D*, vol. 91, p. 075003, 2015.

[24] F. Arbabifar, S. Bahrami, and M. Frank, "Neutral Higgs Bosons in the Higgs Triplet Model with nontrivial mixing," *Phys. Rev. D*, vol. 87, no. 1, p. 015020, 2013.

[25] M. Gell-Mann, "Isotopic spin and new unstable particles," *Phys. Rev.*, vol. 92, pp. 833–834, Nov 1953.

[26] J.-Y. Cen, J.-H. Chen, X.-G. He, G. Li, J.-Y. Su, and W. Wang, "Searching for a charged higgs boson with both $HW^{\pm}Z$ and $H^{\pm}tb$ couplings at the LHC," *Journal of High Energy Physics*, vol. 2019, p. 148, Jan 2019.

[27] V. Brdar, I. Picek, and B. Radovcic, "Radiative Neutrino Mass with Scotogenic Scalar Triplet," *Phys. Lett. B*, vol. 728, pp. 198–201, 2014.

[28] P. Fileviez Pérez, H. H. Patel, M. J. Ramsey-Musolf, and K. Wang, "Triplet scalars and dark matter at the lhc," *Phys. Rev. D*, vol. 79, p. 055024, Mar 2009.

[29] T. P. Cheng and L.-F. Li, "Neutrino masses, mixings, and oscillations in su(2)x u(1) models of electroweak interactions," *Phys. Rev. D*, vol. 22, pp. 2860–2868, Dec 1980.

[30] S. Rappoccio, "The experimental status of direct searches for exotic physics beyond the standard model at the Large Hadron Collider," *Rev. Phys.*, vol. 4, p. 100027, 2019.

[31] R. A. Diaz, R. Martinez, and F. Ochoa, "The Scalar sector of the SU(3)(c) x SU(3)(L) x U(1)(X) model," *Phys. Rev. D*, vol. 69, p. 095009, 2004.

[32] R. Foot, O. F. Hernandez, F. Pisano, and V. Pleitez, "Lepton masses in an SU(3)-L x U(1)-N gauge model," *Phys. Rev. D*, vol. 47, pp. 4158–4161, 1993.

[33] M. Tully and G. C. Joshi, "The Scalar sector in 331 models," *Int. J. Mod. Phys. A*, vol. 18, pp. 1573–1586, 2003.

[34] N. A. Ky and N. T. H. Van, "Scalar sextet in the 331 model with right-handed neutrinos," *Phys. Rev. D*, vol. 72, p. 115017, 2005.

[35] F. Pisano and V. Pleitez, "An SU(3) x U(1) model for electroweak interactions," *Phys. Rev. D*, vol. 46, pp. 410–417, 1992.

[36] P. Frampton, "Chiral dilepton model and the flavor question," *Phys. Rev. Lett.*, vol. 69, pp. 2889–2891, 1992.

[37] M. Singer, J. Valle, and J. Schechter, "Canonical Neutral Current Predictions From the Weak Electromagnetic Gauge Group SU(3) X u(1)," *Phys. Rev. D*, vol. 22, p. 738, 1980.

[38] J. Montero, F. Pisano, and V. Pleitez, "Neutral currents and GIM mechanism in SU(3)-L x U(1)-N models for electroweak interactions," *Phys. Rev. D*, vol. 47, pp. 2918–2929, 1993.

[39] R. Foot, H. N. Long, and T. A. Tran, "$SU(3)_L \otimes U(1)_N$ and $SU(4)_L \otimes U(1)_N$ gauge models with right-handed neutrinos," *Phys. Rev. D*, vol. 50, no. 1, pp. 34–38, 1994.

[40] A. E. C. Hernández, H. N. Long, M. L. Mora-Urrutia, N. H. Thao, and V. V. Vien, "Fermion masses and mixings and $g-2$ muon anomaly in a 3-3-1 model with d_4 family symmetry," 2021.

[41] R. Slansky, "Group Theory for Unified Model Building," *Phys. Rept.*, vol. 79, pp. 1–128, 1981.

[42] G. Senjanovic and R. N. Mohapatra, "Exact left-right symmetry and spontaneous violation of parity," *Phys. Rev. D*, vol. 12, pp. 1502–1505, Sep 1975.

[43] R. N. Mohapatra and G. Senjanovic, "Neutrino Masses and Mixings in Gauge Models with Spontaneous Parity Violation," *Phys. Rev.*, vol. D23, p. 165, 1981.

[44] H. Diaz, E. Castillo-Ruiz, O. Pereyra Ravinez, and V. Pleitez, "Explicit parity violation in SU(2)$_L$⊗ SU(2)$_R$⊗ U(1)$_{B-L}$ models," *J. Phys. G*, vol. 48, no. 8, p. 085010, 2021.

[45] R. N. Mohapatra and G. Senjanovic, "Neutrino Mass and Spontaneous Parity Nonconservation," *Phys. Rev. Lett.*, vol. 44, p. 912, 1980. [,231(1979)].

[46] Y. Zhang, H. An, X. Ji, and R. N. Mohapatra, "General CP Violation in Minimal Left-Right Symmetric Model and Constraints on the Right-Handed Scale," *Nucl. Phys.*, vol. B802, pp. 247–279, 2008.

[47] N. G. Deshpande, J. F. Gunion, B. Kayser, and F. I. Olness, "Left-right symmetric electroweak models with triplet Higgs," *Phys. Rev. D*, vol. 44, pp. 837–858, 1991.

[48] A. Maiezza, G. Senjanović, and J. C. Vasquez, "Higgs sector of the minimal left-right symmetric theory," *Phys. Rev. D*, vol. 95, no. 9, p. 095004, 2017.

[49] R. N. Mohapatra and R. E. Marshak, "Local $b - l$ symmetry of electroweak interactions, majorana neutrinos, and neutron oscillations," *Phys. Rev. Lett.*, vol. 44, pp. 1316–1319, May 1980.

[50] W. Heisenberg, "On the structure of atomic nuclei," *Z. Phys.*, vol. 77, pp. 1–11, 1932.